

AN EQUIVARIANT MAIN CONJECTURE IN IWASAWA THEORY AND APPLICATIONS

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ABSTRACT. We construct a new class of Iwasawa modules, which are the number field analogues of the p -adic realizations of the Picard 1-motives constructed by Deligne in [6] and studied extensively from a Galois module structure point of view in our previous work [13] and [12]. We prove that the new Iwasawa modules are of projective dimension 1 over the appropriate profinite group rings. In the abelian case, we prove an Equivariant Main Conjecture, identifying the first Fitting ideal of the Iwasawa module in question over the appropriate profinite group ring with the principal ideal generated by a certain equivariant p -adic L -function. This is an integral, equivariant refinement of the classical Main Conjecture over totally real number fields proved by Wiles in [34]. Finally, we use these results and Iwasawa co-descent to prove refinements of the (imprimitive) Brumer-Stark Conjecture and the Coates-Sinnott Conjecture, away from their 2-primary components, in the most general number field setting. All of the above is achieved under the assumption that the relevant prime p is odd and that the appropriate classical Iwasawa μ -invariants vanish (as conjectured by Iwasawa.)

1. Introduction

Let us consider the data $(\mathcal{K}, \mathcal{S}, \mathcal{T}, p)$, where p is an odd prime, \mathcal{K} is the cyclotomic \mathbb{Z}_p -extension of a number field K (i.e. \mathcal{K} is a \mathbb{Z}_p -field, in the terminology of [15]), and \mathcal{S} and \mathcal{T} are two finite sets of finite primes in \mathcal{K} . We assume that $\mathcal{T} \cap (\mathcal{S} \cup \mathcal{S}_p) = \emptyset$, where \mathcal{S}_p is the set of p -adic primes in \mathcal{K} . Also, let us assume that Iwasawa's μ -invariant conjecture holds for K and p , i.e. we have $\mu_{K,p} = 0$, where $\mu_{K,p}$ is the classical Iwasawa μ -invariant associated to K and p .

The main goals of this paper are threefold:

Firstly, for any data $(\mathcal{K}, \mathcal{S}, \mathcal{T}, p)$ as above, we construct a new class of Iwasawa modules $T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}})$ which are \mathbb{Z}_p -free of finite rank (see §3 below.) We study the $\mathbb{Z}_p[[\mathcal{G}]]$ -module structure of these modules, under the assumption that \mathcal{K} is of CM-type, \mathcal{K} is a Galois extension of a totally real number field k of Galois group $\mathcal{G} := \text{Gal}(\mathcal{K}/k)$, the sets \mathcal{S} and \mathcal{T} are \mathcal{G} -invariant, $\mathcal{T} \neq \emptyset$ and \mathcal{S} contains the finite primes which ramify in \mathcal{K}/k . In this context, we prove that

$$(1) \quad \text{pd}_{\mathbb{Z}_p[[\mathcal{G}]]-T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}})^{-}} = 1,$$

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whenever $T_p(\mathcal{M}_{S,\tau}^\mathcal{K})^- \neq 0$. (See Theorem 4.6 below.) If \mathcal{G} is abelian, then the equality above implies that the first Fitting ideal $\text{Fit}_{\mathbb{Z}_p[[\mathcal{G}]]^-} T_p(\mathcal{M}_{S,\tau}^\mathcal{K})^-$ is principal (see Proposition 7.2(1) below.) This brings us naturally to our next goal, namely the construction of a canonical generator of this ideal.

Secondly, working under the additional assumption that \mathcal{G} is abelian, we construct an equivariant p -adic L -function $\Theta_{S,T}^{(\infty)} \in \mathbb{Z}_p[[\mathcal{G}]]^-$ (see §5.2 below) and prove the following Equivariant Main Conjecture - type equality.

$$(2) \quad \text{Fit}_{\mathbb{Z}_p[[\mathcal{G}]]^-} T_p(\mathcal{M}_{S,\tau}^\mathcal{K})^- = \Theta_{S,T}^{(\infty)} \cdot \mathbb{Z}_p[[\mathcal{G}]]^-.$$

(See Theorem 5.6 below.) This refines the classical Main Conjecture proved by Wiles in [34] in the following precise sense.

For simplicity, let us assume that \mathcal{K} contains the group μ_{p^∞} of p -power roots of unity. Then, under our working assumption that $\mu_{K,p} = 0$, Wiles's main theorem in [34] can be restated as follows.

$$(3) \quad \text{Fit}_{\mathbb{Q}_p(\mathbb{Z}_p[[\mathcal{G}]]^-)} (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathfrak{X}_S^+(-1)^*) = \mathfrak{G}_S^{(\infty)} \cdot \mathbb{Q}_p(\mathbb{Z}_p[[\mathcal{G}]]^-),$$

where $\mathbb{Q}_p(\mathbb{Z}_p[[\mathcal{G}]]^-) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\mathcal{G}]]^-$, \mathfrak{X}_S is the classical Iwasawa module given by the Galois group of the maximal pro- p abelian extension of \mathcal{K} , unramified away from $\mathcal{S} \cup \mathcal{S}_p$, and $\mathfrak{G}_S^{(\infty)}$ is a canonical element¹ in $\mathbb{Q}_p(\mathbb{Z}_p[[\mathcal{G}]]^-)$ constructed out of the S -imprimitive p -adic L -functions associated to \mathcal{K}/k .

A naïve integral refinement of Wiles's result would aim for a calculation of $\text{Fit}_{\mathbb{Z}_p[[\mathcal{G}]]^-} (\mathfrak{X}_S^+(-1)^*)$ in terms of p -adic L -functions. Unfortunately, the $\mathbb{Z}_p[[\mathcal{G}]]^-$ -module $\mathfrak{X}_S^+(-1)^*$ is only very rarely of finite projective dimension. Consequently, its Fitting ideal is not principal and difficult to compute, in general. However, we prove that there is an exact sequence of $\mathbb{Z}_p[[\mathcal{G}]]^-$ -modules

$$(4) \quad 0 \longrightarrow T_p(\Delta_{\mathcal{K},\tau})^- / \mathbb{Z}_p(1) \longrightarrow T_p(\mathcal{M}_{S,\tau}^\mathcal{K})^- \longrightarrow \mathfrak{X}_S^+(-1)^* \longrightarrow 0.$$

(See §3.3 below for the proof and notations.) So, the new modules $T_p(\mathcal{M}_{S,\tau}^\mathcal{K})^-$ are certain extensions of the classical modules $\mathfrak{X}_S^+(-1)^*$ by $T_p(\Delta_{\mathcal{K},\tau})^- / \mathbb{Z}_p(1)$. These modules happen to be of projective dimension 1 over $\mathbb{Z}_p[[\mathcal{G}]]^-$. Also, a consequence of the definition of the equivariant p -adic L -function $\Theta_{S,T}^{(\infty)}$ is that

$$(5) \quad \Theta_{S,T}^{(\infty)} = \mathfrak{G}_S^{(\infty)} \cdot \mathfrak{D}_T^{(\infty)},$$

where $\mathfrak{D}_T^{(\infty)}$ is a non zero-divisor distinguished generator² of the Fitting ideal $\text{Fit}_{\mathbb{Q}_p(\mathbb{Z}_p[[\mathcal{G}]]^-)} (T_p(\Delta_{\mathcal{K},\tau})^- / \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$. Since Fitting ideals over $\mathbb{Q}_p(\mathbb{Z}_p[[\mathcal{G}]]^-)$ are multiplicative in short exact sequences, by way of (4)–(5) our main theorem (2) “tensoring with \mathbb{Q}_p ” (i.e. base-changed from the integral group ring $\mathbb{Z}_p[[\mathcal{G}]]^-$ to the rational one $\mathbb{Q}_p(\mathbb{Z}_p[[\mathcal{G}]]^-)$) is equivalent to Wiles's main theorem (3).

Finally, we use our main theorems (1) and (2) and Iwasawa co-descent to prove refinements of the (imprimitive) Brumer-Stark Conjecture (see Theorem 6.5 below) and the Coates-Sinnott Conjecture (see Theorem 6.11 and Corollary 6.19 below), away from their 2-primary component, in the most general number field setting.

¹The reader can easily check that $\mathfrak{G}_S^{(\infty)} = (\iota \circ t_1)(G_S)$, with notations as in §5.2.

²The reader can easily check that $\mathfrak{D}_T^{(\infty)} = \delta_T^{(\infty)} / (\iota \circ t_1)(H_S)$, with notations as in §5.2.

The main ideas and motivation behind the construction of the new Iwasawa modules $T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}})$ are rooted in Deligne's theory of 1-motives (see [6]) and in our previous work [13] on the Galois module structure of p -adic realizations of Picard 1-motives. Also, we were strongly motivated and inspired by the Deligne-Tate proof of the Brumer-Stark Conjecture in function fields via p -adic realizations of Picard 1-motives. (See Chapter V of [32].)

In [13], we consider the Picard 1-motive $\mathcal{M}_{\mathcal{S},\mathcal{T}}^X$ associated to a smooth, projective curve X defined over an arbitrary algebraically closed field κ and two finite, disjoint sets of closed points \mathcal{S} and \mathcal{T} on X . We study the Galois module structure of its p -adic realizations $T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^X)$ in finite G -Galois covers $X \rightarrow Y$ of smooth, projective curves over κ , in the case where \mathcal{S} contains the ramification locus of the cover and \mathcal{S} and \mathcal{T} are G -invariant. In particular, if $\kappa = \overline{\mathbb{F}_q}$, for some finite field \mathbb{F}_q , and the data $(X \rightarrow Y, \mathcal{S}, \mathcal{T})$ is defined over \mathbb{F}_q , with $Y = Y_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$, for a smooth, projective curve Y_0 over \mathbb{F}_q , then we prove that

$$(6) \quad \mathrm{pd}_{\mathbb{Z}_p[[\mathcal{G}]]} T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^X) = 1, \quad \mathrm{Fit}_{\mathbb{Z}_p[[\mathcal{G}]]} T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^X) = \mathbb{Z}_p[[\mathcal{G}]] \cdot \Theta_{\mathcal{S},\mathcal{T}}(\gamma^{-1}),$$

for all primes p , where $\mathcal{G} := \mathrm{Gal}(\overline{\mathbb{F}_q}(X)/\mathbb{F}_q(Y_0))$, γ is the q -power arithmetic Frobenius (viewed in \mathcal{G}) and $\Theta_{\mathcal{S},\mathcal{T}}(\gamma^{-1})$ is an equivariant L -function which is the exact function field analogue of $\Theta_{\mathcal{S},\mathcal{T}}^{(\infty)}$. Theorem 4.6 (equality (1) above) is the number field analogue of the first equality in (6), while Theorem 5.6 (equality (2) above) is the number field analogue of the second equality in (6). Wiles's Theorem (equality (3) above) is equivalent to the number field analogue of Théorème 2.5 in Ch.V of [32]. The second equality in (6) is a natural integral refinement of Théorème 2.5 in loc.cit. Further, in [13], we use the second equality in (6) and Galois co-descent with respect to $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ to prove refinements of the Brumer-Stark and Coates-Sinnott Conjectures in function fields. In the number field case, these tasks are achieved with almost identical techniques, in Theorems 6.5 and 6.11 below.

In §2.1 below, we define a category of purely algebraic objects which we call “abstract 1-motives”. This category is endowed with covariant functors T_p to the category of free \mathbb{Z}_p -modules of finite rank, called p -adic realizations, for all primes p . Two particular examples of abstract 1-motives are Deligne's Picard 1-motives $\mathcal{M}_{\mathcal{S},\mathcal{T}}^X$ and their number field analogues $\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}}$, constructed in §3.1. The Iwasawa module $T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}})$ of interest in this paper is the p -adic realization of $\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}}$, and it is the number field analogue of $T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^X)$.

The paper is organized as follows. In §2, we define the category of abstract 1-motives and lay down the number theoretic groundwork necessary for constructing the abstract 1-motive $\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}}$. In §3, we construct the abstract 1-motive $\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}}$, show that its p -adic realizations $T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}})$ have a natural Iwasawa module structure and link these to the classical Iwasawa modules $\mathfrak{X}_{\mathcal{S}}^{\pm}(-1)^*$, under the appropriate hypotheses. In §4, we prove the projective dimension result, (1) above. In §5, we prove the Equivariant Main Conjecture, (2) above. In §6, we use the results obtained in the previous sections to prove refined version of the (imprimitive) Brumer-Stark Conjecture and the Coates-Sinnott Conjecture, away from their 2-primary components, in the general number field setting. The paper has an Appendix (§7), where we gather various homological algebra results needed throughout.

In upcoming work, we will give further applications of the results obtained in this paper and its function field companion [13] in the following directions: **I.** An explicit construction of ℓ -adic models for Tate sequences in the general case of global fields; **II.** A proof of the Equivariant Tamagawa Number Conjecture for Dirichlet motives, in full generality for function fields, and on the “minus side” and away from its 2-primary part for CM-extensions of totally real number fields. A proof of the Gross-Rubin-Stark Conjecture (a vast refined generalization of the Brumer-Stark Conjecture, see Conjecture 5.3.4 in [24]) will ensue in the cases listed above; **III.** Finally, we will consider non-abelian versions of the Equivariant Main Conjectures proved in this paper and [13].

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2. Algebraic and number theoretic preliminaries

2.1. Abstract 1-motives. Assume that J is an arbitrary abelian group, m is a strictly positive integer and p is a prime. We denote by $J[m]$ the maximal m -torsion subgroup of J and let $J[p^\infty] := \cup_m J[p^m]$. As usual, $T_p(J)$ will denote the p -adic Tate module of J . By definition, $T_p(J)$ is the \mathbb{Z}_p -module given by

$$T_p(J) := \varprojlim_n J[p^n],$$

where the projective limit is taken with respect to the multiplication-by- p maps. There is an obvious canonical isomorphism of \mathbb{Z}_p -modules

$$T_p(J) \simeq \text{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, J) = \text{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, J[p^\infty]).$$

Now, let us assume that J is an abelian, divisible group. J is said to be of *finite local corank* if there exists a positive integer $r_p(J)$ and a \mathbb{Z}_p -module isomorphism

$$J[p^\infty] \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{r_p(J)},$$

for any prime p . The integer $r_p(J)$ is called the p -corank of J . Obviously, in this case, we have $T_p(J) \simeq \mathbb{Z}_p^{r_p(J)}$, for all primes p . Further, one can use the “Hom”-description of $T_p(J)$ given above combined with the injectivity of the \mathbb{Z}_p -module $J[p^\infty]$ to establish canonical \mathbb{Z}_p -module isomorphisms

$$T_p(J)/p^n T_p(J) \simeq J[p^n],$$

for all primes p and all $n \in \mathbb{Z}_{\geq 1}$. (See Remark 3.8 in [13] for these isomorphisms.)

Definition 2.1. An abstract 1-motive $\mathcal{M} := [L \xrightarrow{\delta} J]$ consists of the following.

- A free \mathbb{Z} -module L of finite rank (also called a lattice in what follows);
- An abelian, divisible group J of finite local corank;
- A group morphism $\delta : L \rightarrow J$.

In many ways, an abstract 1-motive \mathcal{M} enjoys the same properties as an abelian, divisible group of finite local corank. Namely, for any $n \in \mathbb{Z}_{\geq 1}$ and any prime p , one can construct an n -torsion group $\mathcal{M}[n]$, a p -divisible p -torsion group of finite

corank $\mathcal{M}[p^\infty]$, and a p -adic Tate module (or p -adic realization) $T_p(\mathcal{M})$, which is \mathbb{Z}_p -free of finite rank. This is done as follows.

For every $n \in \mathbb{Z}_{\geq 1}$, we take the fiber-product of groups $J \times_J^n L$, with respect to the map $L \xrightarrow{\delta} J$ and the multiplication by n map $J \xrightarrow{n} J$. An element in this fiber product consists of a pair $(j, \lambda) \in J \times L$, such that $nj = \delta(\lambda)$. Since J is a divisible group, the map $J \xrightarrow{n} J$ is surjective. Consequently, we have a commutative diagram (in the category of abelian groups) whose rows are exact.

$$\begin{array}{ccccccc} 0 & \longrightarrow & J[n] & \longrightarrow & J \times_J^n L & \longrightarrow & L \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow \delta \\ 0 & \longrightarrow & J[n] & \longrightarrow & J & \xrightarrow{n} & J \longrightarrow 0 \end{array}$$

Definition 2.2. The group $\mathcal{M}[n]$ of n -torsion points of \mathcal{M} is defined by

$$\mathcal{M}[n] := (J \times_J^n L) \otimes \mathbb{Z}/n\mathbb{Z}.$$

Since L is a free \mathbb{Z} -module, the rows of the above diagram stay exact when tensored with $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$, respectively. Consequently, we have commutative diagrams with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & J[m] & \longrightarrow & \mathcal{M}[m] & \longrightarrow & L \otimes \mathbb{Z}/m\mathbb{Z} \longrightarrow 0 \\ & & \downarrow m/n & & \downarrow & & \downarrow \\ 0 & \longrightarrow & J[n] & \longrightarrow & \mathcal{M}[n] & \longrightarrow & L \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow 0, \end{array}$$

for all $n, m \in \mathbb{N}$, with $n \mid m$, where the left vertical map is multiplication by m/n , the right vertical map is the canonical surjection and the middle vertical map is the unique morphism which makes the diagram commute. This middle vertical morphism maps $(j, \lambda) \otimes \hat{1}$ to $(m/n \cdot j, \lambda) \otimes \hat{1}$ and, in what follows, we refer to it as the multiplication-by- m/n map $\mathcal{M}[m] \xrightarrow{m/n} \mathcal{M}[n]$.

Definition 2.3. For a prime p , the p -adic Tate module $T_p(\mathcal{M})$ of \mathcal{M} is given by

$$T_p(\mathcal{M}) = \varprojlim_n \mathcal{M}[p^n],$$

where the projective limit is taken with respect to the surjective multiplication-by- p maps described above.

This way, for every prime p , we obtain exact sequences of free \mathbb{Z}_p -modules

$$0 \longrightarrow T_p(J) \longrightarrow T_p(\mathcal{M}) \longrightarrow L \otimes \mathbb{Z}_p \longrightarrow 0.$$

Clearly, we have an isomorphism of \mathbb{Z}_p -modules $T_p(\mathcal{M}) \simeq \mathbb{Z}_p^{(r_p(J) + \text{rk}_{\mathbb{Z}} L)}$, where $r_p(J)$ is the p -corank of J and $\text{rk}_{\mathbb{Z}} L$ is the \mathbb{Z} -rank of L .

For $n, m \in \mathbb{Z}_{\geq 1}$ with $n \mid m$, one also has commutative diagrams with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & J[m] & \longrightarrow & \mathcal{M}[m] & \longrightarrow & L \otimes \mathbb{Z}/m\mathbb{Z} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & J[n] & \longrightarrow & \mathcal{M}[n] & \longrightarrow & L \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow 0, \end{array}$$

with injective vertical morphisms described as follows: the left-most morphism is the usual inclusion; the right-most morphism sends $\lambda \otimes \widehat{x}$ to $\lambda \otimes \widehat{m/n \cdot x}$; the middle morphism sends $(j, \lambda) \otimes \widehat{1}$ to $(j, m/n \cdot \lambda) \otimes \widehat{1}$. Now, if p is a prime number, we define

$$\mathcal{M}[p^\infty] := \varinjlim_m \mathcal{M}[p^m],$$

where the inductive limit is taken with respect to the injective maps described above. Obviously, $\mathcal{M}[p^\infty]$ is a torsion, p -divisible group which sits in the middle of an exact sequence

$$0 \longrightarrow J[p^\infty] \longrightarrow \mathcal{M}[p^\infty] \longrightarrow L \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0,$$

and whose p -corank equals $(r_p(J) + \text{rk}_{\mathbb{Z}} L)$. It is not difficult to see that there are canonical \mathbb{Z}_p -module isomorphisms

$$(7) \quad T_p(\mathcal{M}) \simeq T_p(\mathcal{M}[p^\infty]), \quad T_p(\mathcal{M})/p^n T_p(\mathcal{M}) \simeq \mathcal{M}[p^n],$$

for all $n \in \mathbb{Z}_{\geq 1}$.

Remark 2.4. For a given prime p , one can define an abstract p -adic 1-motive $\mathcal{M} := [L \xrightarrow{\delta} J]$ to consist of a \mathbb{Z}_p -module morphism δ between a free \mathbb{Z}_p -module of finite rank L and a p -divisible \mathbb{Z}_p -module of finite corank J . Obviously, the constructions above lead to a p -divisible group $\mathcal{M}[p^\infty]$ and a p -adic Tate module $T_p(\mathcal{M}) \simeq T_p(\mathcal{M}[p^\infty])$, for any abstract p -adic 1-motive \mathcal{M} . Any abstract 1-motive $\mathcal{M} := [L \xrightarrow{\delta} J]$ gives an abstract p -adic 1-motive

$$\mathcal{M}_p := [L \otimes \mathbb{Z}_p \xrightarrow{\delta \otimes 1} J \otimes \mathbb{Z}_p],$$

for every prime p . It is easy to show that there are canonical isomorphisms

$$\mathcal{M}[p^n] \simeq \mathcal{M}_p[p^n], \quad T_p(\mathcal{M}) \simeq T_p(\mathcal{M}_p),$$

for all primes p and all $n \in \mathbb{Z}_{\geq 1}$.

Definition 2.5. A morphism $f : \mathcal{M}' \rightarrow \mathcal{M}$ between two abstract (respectively, abstract p -adic) 1-motives $\mathcal{M}' := [L' \xrightarrow{\delta'} J']$ and $\mathcal{M} := [L \xrightarrow{\delta} J]$ is a pair $f := (\lambda, \iota)$ of group (respectively, \mathbb{Z}_p -module) morphisms $\lambda : L' \rightarrow L$ and $\iota : J' \rightarrow J$, such that $\iota \circ \delta' = \delta \circ \lambda$.

Now, it is clear what it is meant by composition of morphisms and the category of abstract (respectively, abstract p -adic) 1-motives. It is easy to see that any morphism $f = (\lambda, \iota)$ as in the definition above induces canonical $\mathbb{Z}/m\mathbb{Z}$ -module and \mathbb{Z}_p -module morphisms

$$f[m] : \mathcal{M}'[m] \rightarrow \mathcal{M}[m], \quad T_p(f) : T_p(\mathcal{M}') \rightarrow T_p(\mathcal{M}),$$

respectively, for all the appropriate m and p . Moreover, the associations $f \rightarrow f[m]$ and $f \rightarrow T_p(f)$ commute with composition of morphisms, so we obtain functors from the category of abstract (p -adic) 1-motives to that of $\mathbb{Z}/m\mathbb{Z}$ -modules and \mathbb{Z}_p -modules, respectively, for all m and p as above. A consequence is the following.

Remark 2.6. If R is a ring, L and J are (say, left) R -modules and the map δ is R -linear, then the groups $\mathcal{M}[m]$, $T_p(\mathcal{M})$, $\mathcal{M}[p^\infty]$ associated to the abstract (p -adic) 1-motive $\mathcal{M} := [L \xrightarrow{\delta} J]$ are naturally endowed with R -module structures. Also, all the above exact sequences can be viewed as exact sequences in the category of

R -modules. This observation will be of particular interest to us in the case where R is either a group algebra or a profinite group algebra with coefficients in \mathbb{Z} or \mathbb{Z}_p .

Example 2.7. In the case where $J := \mathcal{A}(\kappa)$ is the group of κ -rational points of a semi-abelian variety \mathcal{A} defined over an algebraically closed field κ and L is an arbitrary lattice then, for any group morphism $\delta : L \rightarrow \mathcal{A}(\kappa)$, the abstract 1-motive $\mathcal{M} := [L \xrightarrow{\delta} J]$ is precisely what Deligne calls a 1-motive in [6].

In particular, let X be a smooth, connected, projective curve defined over an algebraically closed field κ , and let \mathcal{S} and \mathcal{T} be two finite, disjoint sets of closed points on X . Let $J_{\mathcal{T}}$ be the generalized Jacobian associated to (X, \mathcal{T}) . We remind the reader that $J_{\mathcal{T}}$ is a semi-abelian variety (an extension of the Jacobian J_X of X by a torus $\tau_{\mathcal{T}}$) defined over κ , whose group of κ -rational points is given by

$$J_{\mathcal{T}}(\kappa) = \frac{\text{Div}^0(X \setminus \mathcal{T})}{\{\text{div}(f) \mid f \in \kappa(X)^\times, f(v) = 1, \forall v \in \mathcal{T}\}}.$$

Here, $\text{Div}^0(X \setminus \mathcal{T})$ denotes the set of X -divisors of degree 0 supported away from \mathcal{T} and $\text{div}(f)$ is the divisor associated to any non-zero element f in the field $\kappa(X)$ of κ -rational functions on X . The group $J_{\mathcal{T}}(\kappa)$ is divisible, as an extension of the divisible groups $J_X(\kappa)$ and $\tau_{\mathcal{T}}(\kappa)$. Its local coranks satisfy

$$p\text{-corank}(J_{\mathcal{T}}(\kappa)) = \begin{cases} 2g_X + |\mathcal{T}| - 1, & p \neq \text{char}(\kappa); \\ \gamma_X, & p = \text{char}(\kappa), \end{cases}$$

where g_X and γ_X are the genus and Hasse–Witt invariant of X , respectively. (See §3 in [13] for the definitions and the above equality.)

If $\text{Div}^0(\mathcal{S})$ is the lattice of divisors of degree 0 on the curve X supported on \mathcal{S} and $\delta : \text{Div}^0(\mathcal{S}) \rightarrow J_{\mathcal{T}}(\kappa)$ is the usual divisor-class map, then the abstract 1-motive

$$\mathcal{M}_{\mathcal{S}, \mathcal{T}}^X := [\text{Div}^0(\mathcal{S}) \xrightarrow{\delta} J_{\mathcal{T}}(\kappa)]$$

is called the Picard 1-motive associated to $(X, \mathcal{S}, \mathcal{T}, \kappa)$. Its ℓ -adic realizations $T_\ell(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^X)$, for all primes ℓ , were extensively studied in [13].

In this paper, we will construct a class of abstract 1-motives which arises naturally in the context of Iwasawa theory of number fields. These should be viewed as the number field analogues of the Picard 1-motives $\mathcal{M}_{\mathcal{S}, \mathcal{T}}^X$ described in the example above. As we will see in the next few sections, their ℓ -adic realizations satisfy similar properties to those proved to be satisfied by $T_\ell(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^X)$ in [13].

2.2. Generalized ideal-class groups. Let p be a prime number. In what follows, we borrow Iwasawa’s terminology (see [15]) and call a field \mathcal{K} a \mathbb{Z}_p -field if it is the cyclotomic \mathbb{Z}_p -extension of a number field. If a \mathbb{Z}_p -field \mathcal{K} contains the group μ_{p^∞} of p -power roots of unity, then it is called a *cyclotomic \mathbb{Z}_p -field*. A \mathbb{Z}_p -field \mathcal{K} is said to be of CM-type if it is the cyclotomic \mathbb{Z}_p -extension of a CM-number field K . This is equivalent to the existence of a (necessarily unique) involution j of \mathcal{K} (the so-called complex conjugation automorphism of \mathcal{K}), such that its fixed subfield $\mathcal{K}^+ := \mathcal{K}^{j=1}$ is the cyclotomic \mathbb{Z}_p -extension of a totally real number field. The uniqueness of j with these properties makes it commute with any field automorphism of \mathcal{K} .

For the moment, \mathcal{K} will denote either a number field or a \mathbb{Z}_p -field, for some fixed prime p . As usual, a (finite) prime in \mathcal{K} is an equivalence class of valuations

on \mathcal{K} . If \mathcal{K} is a number field, all these valuations are discrete of rank one (with value group isomorphic to $(\mathbb{Z}, +)$ with the usual order.) If \mathcal{K} is a \mathbb{Z}_p -field, then its valuations are either discrete of rank one, if they extend an ℓ -adic valuation of \mathbb{Q} , for some prime $\ell \neq p$, or they have value group isomorphic to $(\mathbb{Z}[1/p], +)$ with the usual order, if they extend the p -adic valuation on \mathbb{Q} . For every \mathcal{K} and v as above, we pick a valuation ord_v in the class v as follows.

- If \mathcal{K} is a number field, then ord_v is the unique valuation in the class v whose strict value group $\Gamma_v := \text{ord}_v(\mathcal{K}^\times)$ satisfies $\Gamma_v = \mathbb{Z}$. Note that if \mathcal{K}/\mathcal{K}' is an extension of number fields and v and v' are finite primes in \mathcal{K} and \mathcal{K}' , with v dividing v' , then we have a commutative diagram

$$(8) \quad \begin{array}{ccc} \mathcal{K}^\times & \xrightarrow{\text{ord}_v} & \Gamma_v \\ \uparrow \subseteq & & \uparrow \times e(v/v') \\ \mathcal{K}'^\times & \xrightarrow{\text{ord}_{v'}} & \Gamma_{v'}, \end{array}$$

where $e(v/v') := [\Gamma_v : \text{ord}_v(\mathcal{K}'^\times)]$ is the usual ramification index.

- If \mathcal{K} is a \mathbb{Z}_p -field, then we denote by v_K the prime sitting below v in any number field K contained in \mathcal{K} . We let $\Gamma_v := \varinjlim_K \Gamma_{v_K}$, where the limit is viewed in the category of ordered groups and is taken with respect to all number fields K contained in \mathcal{K} and the (ordered group morphisms given by) multiplication by the ramification index maps in the diagram above. Then, since $\mathcal{K}^\times = \varinjlim_K K^\times$, where the limit is taken with respect to the inclusion morphisms, we can define

$$\text{ord}_v : \mathcal{K}^\times \longrightarrow \Gamma_v, \quad \text{ord}_v := \varinjlim_K \text{ord}_{v_K}.$$

It is easy to check that ord_v is a valuation on \mathcal{K} in the class v and that $\Gamma_v = \text{ord}_v(\mathcal{K}^\times)$. As a consequence of the existence of a number field K , with $K \subseteq \mathcal{K}$, such that \mathcal{K}/K is a \mathbb{Z}_p -extension totally ramified at all p -adic primes v in K and unramified everywhere else, there are non-canonical ordered group isomorphisms

$$\Gamma_v \simeq \mathbb{Z}[1/p], \quad \Gamma_v \simeq \mathbb{Z},$$

respectively, depending on whether v sits above the p -adic prime in \mathbb{Q} or not. Also, for any (necessarily finite) extension \mathcal{K}/\mathcal{K}' of \mathbb{Z}_p -fields, we have a commutative diagram identical to (8) above.

For any field \mathcal{K} as above, its (additive) divisor group is given by

$$\text{Div}_{\mathcal{K}} := \bigoplus_v \Gamma_v \cdot v,$$

where the direct sum is taken with respect to all the finite primes in \mathcal{K} . Let \mathcal{T} be a finite (possibly empty) set of finite primes of \mathcal{K} , which is disjoint from the set \mathcal{S}_p consisting of all the primes in \mathcal{K} extending the p -adic valuation of \mathbb{Q} . We let

$$\text{Div}_{\mathcal{K}, \mathcal{T}} := \bigoplus_{v \notin \mathcal{T}} \Gamma_v \cdot v, \quad \mathcal{K}_{\mathcal{T}}^\times := \{x \in \mathcal{K}^\times \mid \text{ord}_v(x-1) > 0, \forall v \in \mathcal{T}\}.$$

The usual divisor map associated to \mathcal{K} induces a group morphism

$$\text{div}_{\mathcal{K}} : \mathcal{K}_{\mathcal{T}}^\times \longrightarrow \text{Div}_{\mathcal{K}, \mathcal{T}}, \quad \text{div}_{\mathcal{K}}(x) = \sum_v \text{ord}_v(x) \cdot v = \sum_{v \notin \mathcal{T}} \text{ord}_v(x) \cdot v,$$

whose kernel is the following subgroup of the group $U_{\mathcal{K}}$ of units in \mathcal{K} .

$$U_{\mathcal{K},\mathcal{T}} := \{u \in U_{\mathcal{K}} \mid \text{ord}_v(u-1) > 0, \text{ for all } v \in \mathcal{T}\}.$$

To \mathcal{K} and \mathcal{T} as above, we associate the following generalized ideal-class group

$$C_{\mathcal{K},\mathcal{T}} := \frac{\text{Div}_{\mathcal{K},\mathcal{T}}}{\text{div}_{\mathcal{K}}(\mathcal{K}_{\mathcal{T}}^{\times})}.$$

If \mathcal{K}/\mathcal{K}' is a finite extension of fields as above and \mathcal{T}' is the set of primes in \mathcal{K}' sitting below primes in \mathcal{T} , then we have an injective group morphism

$$\text{Div}_{\mathcal{K}',\mathcal{T}'} \longrightarrow \text{Div}_{\mathcal{K},\mathcal{T}}, \quad v' \rightarrow \sum_{v|v'} e(v/v') \cdot v,$$

where the sum is taken over all the primes v of \mathcal{K} sitting above the prime v' in \mathcal{K}' . This morphism is compatible with the divisor maps (see diagram (8)) and it induces a (not necessarily injective) group morphism $C_{\mathcal{K}',\mathcal{T}'} \rightarrow C_{\mathcal{K},\mathcal{T}}$ at the level of generalized ideal class-groups. It is easy to check that

$$\varinjlim_K \text{Div}_{K,T_K} \simeq \text{Div}_{\mathcal{K},\mathcal{T}}, \quad \varinjlim_K C_{K,T_K} \simeq C_{\mathcal{K},\mathcal{T}},$$

where the limits are taken with respect to all number fields $K \subseteq \mathcal{K}$ and T_K is the set of primes in K sitting below primes in \mathcal{T} . In particular, let us assume that \mathcal{K} is the \mathbb{Z}_p -cyclotomic extension of a number field K and write it as a union of number fields $\mathcal{K} = \cup_n K_n$, where K_n is the unique intermediate field $K \subseteq K_n \subseteq \mathcal{K}$ with $[K_n : K] = p^n$. Then, since the set $\{K_n \mid n \geq 1\}$ is cofinal (with respect to inclusion) in the set of all number fields contained in \mathcal{K} , we have group isomorphisms

$$(9) \quad \varinjlim_n \text{Div}_{K_n,T_n} \simeq \text{Div}_{\mathcal{K},\mathcal{T}}, \quad \varinjlim_n C_{K_n,T_n} \simeq C_{\mathcal{K},\mathcal{T}},$$

where T_n is the set of primes in K_n sitting below those in \mathcal{T} and the injective limits are taken with respect to the morphisms described above.

If $\mathcal{T} = \emptyset$, then we drop it from the notation arriving this way at the classical group of units $U_{\mathcal{K}}$ and ideal-class group $C_{\mathcal{K}}$ associated to \mathcal{K} . Let

$$\Delta_{\mathcal{K},\mathcal{T}} := \bigoplus_{v \in \mathcal{T}} \kappa(v)^{\times},$$

where $\kappa(v)$ denotes the residue field associated to the prime v . It is easy to see that for any \mathcal{T} as above we have an exact sequence of groups

$$(10) \quad 0 \longrightarrow U_{\mathcal{K}}/U_{\mathcal{K},\mathcal{T}} \longrightarrow \Delta_{\mathcal{K},\mathcal{T}} \longrightarrow C_{\mathcal{K},\mathcal{T}} \longrightarrow C_{\mathcal{K}} \longrightarrow 0.$$

If \mathcal{K} is a number field, the exact sequence above is described in detail in [26] or [24], for example. If \mathcal{K} is the cyclotomic \mathbb{Z}_p -extension of a number field K , then the exact sequence above is obtained by taking the injective limit of the corresponding sequences at the finite levels K_n with respect to the obvious transition maps. If \mathcal{K} is a number field, (10) shows that $C_{\mathcal{K},\mathcal{T}}$ is finite. In that case, the Artin reciprocity map associated to \mathcal{K} establishes an isomorphism between $C_{\mathcal{K},\mathcal{T}}$ and the Galois group of the maximal abelian extension of \mathcal{K} which is unramified away from \mathcal{T} and at most tamely ramified at primes in \mathcal{T} (see [26].) In the case of a \mathbb{Z}_p -field \mathcal{K} , the second isomorphism in (9) shows that $C_{\mathcal{K},\mathcal{T}}$ is a torsion group.

Throughout the rest of this section, \mathcal{K} is a \mathbb{Z}_p -field, for some prime p . We let

$$\mathcal{A}_{\mathcal{K}} := C_{\mathcal{K}} \otimes \mathbb{Z}_p, \quad \mathcal{A}_{\mathcal{K}, \mathcal{T}} := C_{\mathcal{K}, \mathcal{T}} \otimes \mathbb{Z}_p,$$

for any \mathcal{T} as above. A classical theorem of Iwasawa (see [15], pp. 272-273 and the references therein) shows that there is an isomorphism of groups

$$\mathcal{A}_{\mathcal{K}} \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{\lambda_{\mathcal{K}}} \oplus \mathcal{A}',$$

where $\lambda_{\mathcal{K}}$ is a positive integer which depends only on \mathcal{K} , called the λ -invariant of \mathcal{K} and \mathcal{A}' is a torsion \mathbb{Z}_p -module of finite exponent. Iwasawa conjectured that \mathcal{A}' is trivial (see loc. cit.). This conjecture is equivalent to the vanishing of the classical Iwasawa μ -invariants $\mu_{\mathcal{K}/K} := \mu_{K,p}$ associated to p and all number fields K such that \mathcal{K} is the cyclotomic \mathbb{Z}_p -extension of K . For a given \mathbb{Z}_p -field \mathcal{K} , although $\mu_{\mathcal{K}/K}$ depends on the chosen K with the above properties, its vanishing is independent of that choice. The vanishing is known to hold if \mathcal{K} is the cyclotomic \mathbb{Z}_p -extension of an abelian number field (see [9].) In what follows, if \mathcal{K} is a \mathbb{Z}_p -field, we write $\mu_{\mathcal{K}} = 0$ to mean that $\mu_{\mathcal{K}/K} = 0$, for all (one) K as above.

Now, assume that \mathcal{K} is of CM-type and j is its complex conjugation automorphism. Assume that the prime p is odd. For any \mathbb{Z} -module M endowed with a j -action (called a j -module in what follows) and on which multiplication by 2 is invertible, we let $M^{\pm} := \frac{1}{2}(1 \pm j) \cdot M$ denote the \pm -eigenspaces of j on M . We have a direct sum decomposition $M = M^- \oplus M^+$ and the two functors $M \rightarrow M^{\pm}$ are obviously exact. In particular, if we take $M := \mathcal{A}_{\mathcal{K}}$, we obtain a direct sum decomposition and \mathbb{Z}_p -module isomorphisms

$$\mathcal{A}_{\mathcal{K}} = \mathcal{A}_{\mathcal{K}}^- \oplus \mathcal{A}_{\mathcal{K}}^+, \quad \mathcal{A}_{\mathcal{K}}^- \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{\lambda_{\mathcal{K}}^-} \oplus \mathcal{A}'^-, \quad \mathcal{A}_{\mathcal{K}}^+ \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{\lambda_{\mathcal{K}}^+} \oplus \mathcal{A}'^+,$$

where $\lambda_{\mathcal{K}}^{\pm}$ are positive integers such that $\lambda_{\mathcal{K}} = \lambda_{\mathcal{K}}^- + \lambda_{\mathcal{K}}^+$, and \mathcal{A}^{\pm} are torsion \mathbb{Z}_p -modules of finite exponent, such that $\mathcal{A}' = \mathcal{A}'^+ \oplus \mathcal{A}'^-$. Also, if the set of primes \mathcal{T} above is j -invariant, then (10) is as an exact sequence in the category of j -modules. Consequently, we obtain two exact sequences of \mathbb{Z}_p -modules:

(11)

$$0 \longrightarrow (U_{\mathcal{K}}/U_{\mathcal{K}, \mathcal{T}} \otimes \mathbb{Z}_p)^{\pm} \longrightarrow (\Delta_{\mathcal{K}, \mathcal{T}} \otimes \mathbb{Z}_p)^{\pm} \longrightarrow \mathcal{A}_{\mathcal{K}, \mathcal{T}}^{\pm} \longrightarrow \mathcal{A}_{\mathcal{K}}^{\pm} \longrightarrow 0.$$

Lemma 2.8. *Let \mathcal{K} be a \mathbb{Z}_p -field and \mathcal{T} a finite, non-empty set of finite primes in \mathcal{K} , disjoint from the set of p -adic primes \mathcal{S}_p . Then the following hold.*

- (1) *The module $\Delta_{\mathcal{K}, \mathcal{T}} \otimes \mathbb{Z}_p$ is p -torsion, divisible, of finite corank denoted $\delta_{\mathcal{K}, \mathcal{T}}$.*
- (2) *If $\mu_{\mathcal{K}} = 0$, then the module $\mathcal{A}_{\mathcal{K}, \mathcal{T}}$ is p -torsion, divisible, of corank at most $\lambda_{\mathcal{K}} + \delta_{\mathcal{K}, \mathcal{T}}$.*
- (3) *If \mathcal{K} is of CM-type, p is odd, \mathcal{T} is j -invariant, and $\mu_{\mathcal{K}} = 0$, then the module $\mathcal{A}_{\mathcal{K}, \mathcal{T}}^-$ is p -torsion, divisible, of corank*

$$\lambda_{\mathcal{K}}^- + \delta_{\mathcal{K}, \mathcal{T}}^- - \delta_{\mathcal{K}},$$

where $\delta_{\mathcal{K}, \mathcal{T}}^{\pm} := \text{corank}(\Delta_{\mathcal{K}, \mathcal{T}} \otimes \mathbb{Z}_p)^{\pm}$ and $\delta_{\mathcal{K}} = 1$, if \mathcal{K} is a cyclotomic \mathbb{Z}_p -field and $\delta_{\mathcal{K}} = 0$, otherwise.

Proof. (1) Assume that \mathcal{K} is the cyclotomic \mathbb{Z}_p -extension of a number field K , let $v \in \mathcal{T}$ and let v_0 be the prime in K sitting below v . Since $v \notin \mathcal{S}_p$, the prime v_0 splits completely up to a finite level K_n in \mathcal{K}/K and remains inert in \mathcal{K}/K_n .

As a consequence, the residue field $\kappa(v)$ associated to v is a \mathbb{Z}_p -extension of the finite field $\kappa(v_0)$. Consequently, the p -primary part $\kappa(v)^\times \otimes \mathbb{Z}_p$ of the torsion group $\kappa(v)^\times$ either equals the group μ_{p^∞} of all the p -power roots of unity or it is trivial, according to whether $\kappa(v_0)^\times$ contains μ_p or not. Consequently,

$$\Delta_{\mathcal{K}, \mathcal{T}} \otimes \mathbb{Z}_p \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{\delta_{\mathcal{K}, \mathcal{T}}},$$

where $\delta_{\mathcal{K}, \mathcal{T}}$ is the number of primes $v \in \mathcal{T}$, such that $\mu_p \subseteq \kappa(v)^\times$. Of course, if \mathcal{K} is a cyclotomic \mathbb{Z}_p -field, then $\delta_{\mathcal{K}, \mathcal{T}} = |\mathcal{T}|$. This is because no non-trivial p -power root of unity is congruent to 1 modulo a non- p -adic prime v , so the reduction mod v maps $\mu_{p^\infty} \rightarrow \kappa(v)$ are injective, for all $v \notin \mathcal{S}_p$, in particular for $v \in \mathcal{T}$.

Part (2) is a consequence of part (1), exact sequence (10), and the fact that the category of p -torsion, divisible, groups of finite corank is closed under quotients and extensions.

Under the hypotheses of part (3), we obviously have

$$(12) \quad (U_{\mathcal{K}} \otimes \mathbb{Z}_p)^- = \mu_{p^\infty} \cap \mathcal{K}^\times \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{\delta_{\mathcal{K}}}.$$

Since $\mathcal{T} \cap \mathcal{S}_p = \emptyset$, this implies that $(U_{\mathcal{K}, \mathcal{T}} \otimes \mathbb{Z}_p)^-$ is trivial. In light of these facts, part (3) is a direct consequence of exact sequence (11). \square

Now, assume that \mathcal{K} is the cyclotomic \mathbb{Z}_p -extension of the number field K and that \mathcal{K} is of CM-type. Then, every intermediate field K_n is a CM-number field whose complex conjugation automorphism is the restriction of the complex conjugation j of \mathcal{K} to K_n and will be denoted by j as well in what follows. For a finite set \mathcal{T} of finite primes in \mathcal{K} , disjoint from \mathcal{S}_p , we denote by T_n the set of primes in K_n sitting below primes in \mathcal{T} , for all $n \geq 0$. We let

$$A_{K_n} := C_{K_n} \otimes \mathbb{Z}_p, \quad A_{K_n, T_n} := C_{K_n, T_n} \otimes \mathbb{Z}_p.$$

If p is odd and the set \mathcal{T} is j -invariant, then we can split these \mathbb{Z}_p -modules

$$A_{K_n} = A_{K_n}^+ \oplus A_{K_n}^-, \quad A_{K_n, T_n} := A_{K_n, T_n}^+ \oplus A_{K_n, T_n}^-,$$

into their \pm -eigenspaces with respect to the j -action.

Lemma 2.9. *Under the above hypotheses, the natural maps*

$$A_{K_n, T_n}^- \longrightarrow A_{K_{n+1}, T_{n+1}}^-$$

are injective, for all $n \geq 0$.

Proof. If $\mathcal{T} = \emptyset$, this is Prop. 13.26 in [33]. The general case follows from the case $\mathcal{T} = \emptyset$ by a snake lemma argument applied to the “minus” exact sequence (11). \square

3. THE RELEVANT IWASAWA MODULES

3.1. An Iwasawa theoretic abstract 1-motive. In this section, we fix a prime p , a \mathbb{Z}_p -field \mathcal{K} and two finite sets \mathcal{S} and \mathcal{T} of finite primes in \mathcal{K} , with the property that $\mathcal{T} \cap (\mathcal{S} \cup \mathcal{S}_p) = \emptyset$. From this point on, we assume that $\mu_{\mathcal{K}} = 0$. To the data $(\mathcal{K}, \mathcal{S}, \mathcal{T})$ we associate the abstract 1-motive

$$\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}} := [\text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p) \xrightarrow{\delta} \mathcal{A}_{\mathcal{K}, \mathcal{T}}],$$

where $\text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p)$ is the group of divisors of \mathcal{K} supported on $\mathcal{S} \setminus \mathcal{S}_p$ and δ is the usual divisor-class map sending the divisor D into $\widehat{D} \otimes 1$ in $\mathcal{A}_{\mathcal{K},\mathcal{T}} = C_{\mathcal{K},\mathcal{T}} \otimes \mathbb{Z}_p$, where \widehat{D} denotes the class of D in $C_{\mathcal{K},\mathcal{T}}$. Note that, under our current hypotheses, $\text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p)$ is a free \mathbb{Z} -module of rank $d_{\mathcal{K},\mathcal{S}} := |\mathcal{S} \setminus \mathcal{S}_p|$ and $\mathcal{A}_{\mathcal{K},\mathcal{T}}$ is torsion, divisible, of finite local corank. This local corank equals 0 at primes $\ell \neq p$ and equals at most $(\lambda_{\mathcal{K}} + \delta_{\mathcal{K},\mathcal{T}})$ at p , as Lemma 2.8(2) shows. Consequently, all the requirements in Definition 2.1 are met. The p -adic realization $T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}})$ of $\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}}$ is a free \mathbb{Z}_p -module of rank at most $(\lambda_{\mathcal{K}} + \delta_{\mathcal{K},\mathcal{T}} + d_{\mathcal{K},\mathcal{S}})$, sitting in an exact sequence

$$0 \longrightarrow T_p(\mathcal{A}_{\mathcal{K},\mathcal{T}}) \longrightarrow T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}}) \longrightarrow \text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p) \otimes \mathbb{Z}_p \longrightarrow 0.$$

Next, we give a new interpretation of the p^n -torsion groups $\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}}[p^n]$ of the abstract 1-motive above, for all $n \geq 1$. For that purpose, we need the following.

Definition 3.1. *For every $p, \mathcal{K}, \mathcal{S}, \mathcal{T}$ as above and every $m := p^n$, where $n \in \mathbb{Z}_{\geq 1}$, we define the following subgroup of $\mathcal{K}_{\mathcal{T}}^{\times}$.*

$$\mathcal{K}_{\mathcal{S},\mathcal{T}}^{(m)} := \{f \in \mathcal{K}_{\mathcal{T}}^{\times} \mid \text{div}_{\mathcal{K}}(f) = mD + y\},$$

where $D \in \text{Div}_{\mathcal{K},\mathcal{T}}$ and $y \in \text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p)$. (Note that since the group $\text{Div}_{\mathcal{K}}(\mathcal{S}_p)$ is p -divisible, we could write $y \in \text{Div}_{\mathcal{K}}(\mathcal{S})$ or $y \in \text{Div}_{\mathcal{K}}(\mathcal{S} \cup \mathcal{S}_p)$ instead of $y \in \text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p)$ and arrive at an equivalent definition for $\mathcal{K}_{\mathcal{S},\mathcal{T}}^{(m)}$.)

Note that we have an inclusion $\mathcal{K}_{\mathcal{T}}^{\times m} \cdot U_{\mathcal{K},\mathcal{T}} \subseteq \mathcal{K}_{\mathcal{S},\mathcal{T}}^{(m)}$ of subgroups of $\mathcal{K}_{\mathcal{T}}^{\times}$.

Proposition 3.2. *For every $p, \mathcal{K}, \mathcal{S}, \mathcal{T}$ and m as in the definition above, we have a canonical group isomorphism*

$$\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}}[m] \simeq \mathcal{K}_{\mathcal{S},\mathcal{T}}^{(m)} / (\mathcal{K}_{\mathcal{T}}^{\times m} \cdot U_{\mathcal{K},\mathcal{T}}).$$

Proof. In what follows, in order to simplify notation, we view $\mathcal{A}_{\mathcal{K},\mathcal{T}}$ as the p -Sylow subgroup of the torsion group $C_{\mathcal{K},\mathcal{T}}$. The general element in the fiber-product $\mathcal{A}_{\mathcal{K},\mathcal{T}} \times_{\mathcal{A}_{\mathcal{K},\mathcal{T}}}^m \text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p)$ (see §2.1 for the definition) consists of a pair (\widehat{D}, x) , where $D \in \text{Div}_{\mathcal{K},\mathcal{T}}$, such that its class \widehat{D} in $C_{\mathcal{K},\mathcal{T}}$ lies in $\mathcal{A}_{\mathcal{K},\mathcal{T}}$, and $x \in \text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p)$ with the property that $mD - x = \text{div}_{\mathcal{K}}(f)$, for some $f \in \mathcal{K}_{\mathcal{T}}^{\times}$. Note that, by definition, any such f lies in $\mathcal{K}_{\mathcal{S},\mathcal{T}}^{(m)}$. We define a map

$$\mathcal{A}_{\mathcal{K},\mathcal{T}} \times_{\mathcal{A}_{\mathcal{K},\mathcal{T}}}^m \text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p) \xrightarrow{\phi} \mathcal{K}_{\mathcal{S},\mathcal{T}}^{(m)} / (\mathcal{K}_{\mathcal{T}}^{\times m} \cdot U_{\mathcal{K},\mathcal{T}}), \quad \phi(\widehat{D}, x) = \widehat{f},$$

where D, x and f are as above, and \widehat{f} is the class of f in the quotient to the right. We claim that ϕ is a well-defined, surjective group morphism. We check this next.

Step 1. ϕ is a well defined group morphism. Assume that $D, D' \in \text{Div}_{\mathcal{K},\mathcal{T}}$ and $x \in \text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p)$, such that (\widehat{D}, x) and (\widehat{D}', x) belong to and are equal in the fiber product above. This means that

$$\text{div}_{\mathcal{K}}(f) = mD - x, \quad \text{div}_{\mathcal{K}}(f') = mD' - x, \quad D - D' = \text{div}_{\mathcal{K}}(g),$$

for some $f, f', g \in \mathcal{K}_{\mathcal{T}}^{\times}$. Obviously, this implies that $\text{div}_{\mathcal{K},\mathcal{T}}(f) = \text{div}_{\mathcal{K},\mathcal{T}}(f'g^m)$. Consequently, there exists $u \in U_{\mathcal{K},\mathcal{T}}$, such that $f = f' \cdot g^m u$. This implies that $\phi(\widehat{D}, x) = \widehat{f} = \widehat{f'} = \phi(\widehat{D'}, x)$, showing that ϕ is well-defined as a function. The fact that ϕ is a group morphism is obvious.

Step 2. ϕ is surjective. Let $f \in \mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(m)}$ and $D \in \text{Div}_{\mathcal{K}, \mathcal{T}}$ and $x \in \text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p)$, such that $\text{div}_{\mathcal{K}}(f) = mD - x$. Since $C_{\mathcal{K}, \mathcal{T}}$ is a torsion group, there exists a natural number a , with $\gcd(a, m) = 1$, such that $a\widehat{D} \in \mathcal{A}_{\mathcal{K}, \mathcal{T}}$. This shows that $(a\widehat{D}, -ax) \in \mathcal{A}_{\mathcal{K}, \mathcal{T}} \times_{\mathcal{A}_{\mathcal{K}, \mathcal{T}}}^m \text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p)$ and $\phi(a\widehat{D}, -ax) = \widehat{f}^a$. Consequently, $\widehat{f}^a \in \text{Im}(\phi)$. However, since $\text{Im}(\phi)$ is a group of exponent dividing m (as a subgroup of $\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(m)} / \mathcal{K}_{\mathcal{T}}^{\times m} \cdot U_{\mathcal{K}, \mathcal{T}}$), it is uniquely a -divisible. This shows that $\widehat{f} \in \text{Im}(\phi)$, which concludes the proof of the surjectivity of ϕ .

Step 3. The kernel of ϕ . Next, we prove that

$$\ker(\phi) = m(\mathcal{A}_{\mathcal{K}, \mathcal{T}} \times_{\mathcal{A}_{\mathcal{K}, \mathcal{T}}}^m \text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p)).$$

Let $(\widehat{D}, x) \in \ker(\phi)$. This means that there exists $g \in \mathcal{K}_{\mathcal{T}}^{\times}$, such that

$$mD - x = \text{div}_{\mathcal{K}}(g^m) = m \cdot \text{div}_{\mathcal{K}}(g).$$

This implies that $x = mx'$, for some $x' \in \text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p)$. However, since $\mathcal{A}_{\mathcal{K}, \mathcal{T}}$ is m -divisible, there exists $D' \in \text{Div}_{\mathcal{K}, \mathcal{T}}$, such that $\widehat{D}' \in \mathcal{A}_{\mathcal{K}, \mathcal{T}}$ and $\widehat{D} = m\widehat{D}'$. This means that $D - mD' = \text{div}_{\mathcal{K}}(f')$, for some $f' \in \mathcal{K}_{\mathcal{T}}^{\times}$. This shows that

$$(\widehat{D}, x) = m(\widehat{D}', x'), \quad mD' - x' = \text{div}_{\mathcal{K}}(gf').$$

Therefore, $\ker(\phi) \subseteq m(\mathcal{A}_{\mathcal{K}, \mathcal{T}} \times_{\mathcal{A}_{\mathcal{K}, \mathcal{T}}}^m \text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p))$. The opposite inclusion is obvious.

Now, we combine Steps 1–3 above with the definition of $\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}}[m]$ (see §2.1) to conclude that the map ϕ factors through a group isomorphism $\tilde{\phi}$

$$\begin{array}{ccc} \mathcal{A}_{\mathcal{K}, \mathcal{T}} \times_{\mathcal{A}_{\mathcal{K}, \mathcal{T}}}^m \text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p) & \xrightarrow{\phi} & \mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(m)} / (\mathcal{K}_{\mathcal{T}}^{\times m} \cdot U_{\mathcal{K}, \mathcal{T}}) \\ \downarrow & \nearrow \tilde{\phi} & \\ \mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}}[m] & \xrightarrow{\sim} & \end{array}$$

This concludes the proof of the proposition. \square

Remark 3.3. If G is a group of field automorphisms of \mathcal{K} and the sets \mathcal{S} and \mathcal{T} are G -equivariant, then $T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}})$ and $\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}}[p^n]$ are endowed with natural $\mathbb{Z}_p[G]$ -module structures (see Remark 2.6.) In this case, it is easily seen that the canonical isomorphism in Proposition 3.2 is $\mathbb{Z}_p[G]$ -linear.

If, in addition, \mathcal{K} is of CM-type, p is odd, and \mathcal{S} and \mathcal{T} are j -invariant, where j is the complex conjugation automorphism of \mathcal{K} , then we can talk about the eigenspaces $T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}})^{\pm}$ and $\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}}[p^n]^{\pm}$, for all $n \geq 1$. Since the actions of G and j commute, these eigenspaces have natural $\mathbb{Z}_p[G]$ -module structures. Moreover, the isomorphism in Proposition 3.2 induces two $\mathbb{Z}_p[G]$ -linear isomorphisms

$$\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}}[m]^{\pm} \simeq (\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(m)} / \mathcal{K}_{\mathcal{T}}^{\times m} \cdot U_{\mathcal{K}, \mathcal{T}})^{\pm},$$

for all m which are powers of p . This follows immediately from the fact that the action of j and G on \mathcal{K} commute, for any G as above.

Corollary 3.4. *If \mathcal{K} is of CM-type, p is odd and \mathcal{S} and \mathcal{T} are j -invariant, then the isomorphism in Proposition 3.2 induces isomorphisms of \mathbb{Z}_p -modules*

$$\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}}[m]^- \simeq (\mathcal{K}_{\mathcal{S},\mathcal{T}}^{(m)} / \mathcal{K}_{\mathcal{T}}^{\times m} \cdot U_{\mathcal{K},\mathcal{T}})^- \simeq \left(\mathcal{K}_{\mathcal{S},\mathcal{T}}^{(m)} / \mathcal{K}_{\mathcal{T}}^{\times m} \right)^-,$$

for all $m := p^n$, where $n \geq 1$. If, in addition, \mathcal{S} and \mathcal{T} are G -invariant, for a group of automorphisms G of \mathcal{K} , then the above isomorphisms are $\mathbb{Z}_p[G]$ -linear.

Proof. Let m be as above. There is an obvious exact sequence of $\mathbb{Z}_p[\langle j \rangle]$ -modules

$$U_{\mathcal{K},\mathcal{T}} / U_{\mathcal{K},\mathcal{T}}^m \longrightarrow \mathcal{K}_{\mathcal{S},\mathcal{T}}^{(m)} / \mathcal{K}_{\mathcal{T}}^{\times m} \longrightarrow \mathcal{K}_{\mathcal{S},\mathcal{T}}^{(m)} / (\mathcal{K}_{\mathcal{T}}^{\times m} \cdot U_{\mathcal{K},\mathcal{T}}) \longrightarrow 0.$$

Now, (12) above shows that $(U_{\mathcal{K},\mathcal{T}} \otimes \mathbb{Z}_p)^- = \mu_{p^\infty} \cap U_{\mathcal{K},\mathcal{T}}$. If $\mathcal{T} \neq \emptyset$, this intersection is trivial, whereas if $\mathcal{T} = \emptyset$, this intersection is either equal to μ_{p^∞} or trivial, depending on whether \mathcal{K} is a cyclotomic \mathbb{Z}_p -field or not. In all these cases, $(U_{\mathcal{K},\mathcal{T}} \otimes \mathbb{Z}_p)^-$ is p -divisible. Consequently, $(U_{\mathcal{K},\mathcal{T}} / U_{\mathcal{K},\mathcal{T}}^m)^- \simeq (U_{\mathcal{K},\mathcal{T}} \otimes \mathbb{Z}_p)^- \otimes \mathbb{Z}/m\mathbb{Z} = 0$. Consequently, the exact sequence above leads to a group isomorphism

$$(\mathcal{K}_{\mathcal{S},\mathcal{T}}^{(m)} / \mathcal{K}_{\mathcal{T}}^{\times m})^- \simeq (\mathcal{K}_{\mathcal{S},\mathcal{T}}^{(m)} / \mathcal{K}_{\mathcal{T}}^{\times m} \cdot U_{\mathcal{K},\mathcal{T}})^-.$$

Now, the first part of the corollary is a direct consequence of Proposition 3.2. The $\mathbb{Z}_p[G]$ -linearity is a consequence of the previous Remark combined with the obvious $\mathbb{Z}_p[G]$ -linearity of the last displayed isomorphism. \square

Remark 3.5. *Assume that \mathcal{K}/\mathcal{K}' is an extension of \mathbb{Z}_p -fields, and \mathcal{S} and \mathcal{T} are sets of primes in \mathcal{K} as above. Let \mathcal{S}' , \mathcal{T}' and \mathcal{S}'_p denote the sets of primes in \mathcal{K}' sitting below primes in \mathcal{S} , \mathcal{T} and \mathcal{S}_p , respectively. Then, the inclusion $\mathcal{K}' \rightarrow \mathcal{K}$ induces natural group morphisms $\text{Div}_{\mathcal{K}'}(\mathcal{S}' \setminus \mathcal{S}'_p) \rightarrow \text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p)$ and $\mathcal{A}_{\mathcal{K}',\mathcal{T}'} \rightarrow \mathcal{A}_{\mathcal{K},\mathcal{T}}$. These lead to a morphism of abstract 1-motives $\mathcal{M}_{\mathcal{S}',\mathcal{T}'}^{\mathcal{K}'} \rightarrow \mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}}$. Since the isomorphisms constructed in Proposition 3.2 are functorial, we get commutative diagrams*

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}}[m] & \xrightarrow{\sim} & \mathcal{K}_{\mathcal{S},\mathcal{T}}^{(m)} / (\mathcal{K}_{\mathcal{T}}^{\times m} \cdot U_{\mathcal{K},\mathcal{T}}) \\ \uparrow & & \uparrow \\ \mathcal{M}_{\mathcal{S}',\mathcal{T}'}^{\mathcal{K}'}[m] & \xrightarrow{\sim} & \mathcal{K}_{\mathcal{S}',\mathcal{T}'}^{(m)} / (\mathcal{K}_{\mathcal{T}'}^{\times m} \cdot U_{\mathcal{K}',\mathcal{T}'}). \end{array}$$

3.2. The ensuing Iwasawa modules. Let us assume that \mathcal{K} is the cyclotomic \mathbb{Z}_p -extension of a number field K . As usual, we let $\Gamma := \text{Gal}(\mathcal{K}/K)$. We pick two finite sets \mathcal{S} and \mathcal{T} of finite primes in \mathcal{K} , such that $\mathcal{T} \cap (\mathcal{S} \cup \mathcal{S}_p) = \emptyset$. We assume that \mathcal{S} and \mathcal{T} are Γ -invariant.

For all $n \in \mathbb{Z}_{\geq 0}$, we let $\Gamma_n := \text{Gal}(\mathcal{K}/K_n)$, where K_n is the unique intermediate field $K \subseteq K_n \subseteq \mathcal{K}$, such that $[K_n : K] = p^n$. Also, we let

$$\Lambda = \mathbb{Z}_p[[\Gamma]] := \varprojlim_n \mathbb{Z}_p[\Gamma/\Gamma_n]$$

denote the p -adic profinite group ring associated to Γ , where the projective limit is taken with respect to the surjections $\Gamma/\Gamma_{n+1} \twoheadrightarrow \Gamma/\Gamma_n$ induced by Galois restriction.

As in §2.2, we denote by T_n , S_n and $S_{p,n}$ the sets of primes in K_n which sit below primes in \mathcal{S} , \mathcal{T} , and \mathcal{S}_p , respectively, for all $n \geq 0$. Obviously, S_n , T_n and

$S_{p,n}$ are Γ/Γ_n -invariant. Consequently, A_{K_n, T_n} and $\text{Div}_{K_n}(S_n \setminus S_{p,n}) \otimes \mathbb{Z}_p$ have canonical $\mathbb{Z}_p[\Gamma/\Gamma_n]$ -module structures. Via the ring morphisms $\Lambda \twoheadrightarrow \mathbb{Z}_p[\Gamma/\Gamma_n]$, they are endowed with canonical Λ -module structures. The natural morphisms

$$A_{K_n, T_n} \longrightarrow A_{K_{n+1}, T_{n+1}}, \quad \text{Div}_{K_n}(S_n \setminus S_{p,n}) \otimes \mathbb{Z}_p \longrightarrow \text{Div}_{K_{n+1}}(S_{n+1} \setminus S_{p,n+1}) \otimes \mathbb{Z}_p$$

are Λ -linear. This leads to canonical Λ -module structures for

$$\mathcal{A}_{\mathcal{K}, \mathcal{T}} \simeq \varinjlim_n A_{K_n, T_n} \quad \text{and} \quad \text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p) \otimes \mathbb{Z}_p \simeq \varinjlim_n (\text{Div}_{K_n}(S_n \setminus S_{p,n}) \otimes \mathbb{Z}_p),$$

as direct limits of Λ -modules with respect to Λ -linear transition maps. This way, $T_p(\mathcal{A}_{\mathcal{K}, \mathcal{T}})$ inherits a canonical Λ -module structure as well.

Lemma 3.6. *With notations as above, the \mathbb{Z}_p -module $T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}})$ can be endowed with a natural Λ -module structure which makes the sequence*

$$0 \longrightarrow T_p(\mathcal{A}_{\mathcal{K}, \mathcal{T}}) \longrightarrow T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}}) \longrightarrow \text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p) \otimes \mathbb{Z}_p \longrightarrow 0$$

exact in the category of Λ -modules.

Proof. Remark 2.4 shows that in defining the torsion groups $\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}}[m]$, with m a power of p , and the p -adic Tate module $T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}})$, we may replace the abstract 1-motive $\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}}$ with the associated abstract p -adic 1-motive

$$(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}})_p := [\text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p) \otimes \mathbb{Z}_p \xrightarrow{\delta \otimes 1} \mathcal{A}_{\mathcal{K}, \mathcal{T}}],$$

where $\delta \otimes 1$ is the extension of δ by \mathbb{Z}_p -linearity to $\text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p) \otimes \mathbb{Z}_p$. (Note that $\mathcal{A}_{\mathcal{K}, \mathcal{T}} \simeq \mathcal{A}_{\mathcal{K}, \mathcal{T}} \otimes \mathbb{Z}_p$.) Now, we obviously have

$$[\text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p) \otimes \mathbb{Z}_p \xrightarrow{\delta \otimes 1} \mathcal{A}_{\mathcal{K}, \mathcal{T}}] = \varinjlim_n [\text{Div}_{K_n}(S_n \setminus S_{p,n}) \otimes \mathbb{Z}_p \xrightarrow{\delta_n \otimes 1} A_{K_n, T_n}]$$

where δ_n is the usual divisor-class map sending the divisor $D \in \text{Div}_{K_n}(S_n \setminus S_{p,n})$ to $\widehat{D} \otimes 1$ in $C_{K_n, T_n} \otimes \mathbb{Z}_p = A_{K_n, T_n}$, with \widehat{D} denoting the class of D in C_{K_n, T_n} . Since the maps $\delta_n \otimes 1$ are $\mathbb{Z}_p[\Gamma/\Gamma_n]$ -linear, they are Λ -linear and therefore $\delta \otimes 1$ is Λ -linear. Remark 2.6 applied to $(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}})_p$ and $R := \Lambda$, implies that $\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}}[m] \simeq (\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}})_p[m]$ can be endowed with a natural Λ -module structure which makes the sequence

$$0 \longrightarrow \mathcal{A}_{\mathcal{K}, \mathcal{T}}[m] \longrightarrow \mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}}[m] \longrightarrow \text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p) \otimes \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

exact in the category of Λ -modules, for all m which are powers of p . The statement in the Lemma is now obtained by taking the projective limit of the exact sequences above for $m = p^a$, with $a \in \mathbb{Z}_{\geq 1}$, with respect to the multiplication-by- p maps, which are clearly Λ -linear. \square

Remark 3.7. *Assume that K/k is a Galois extension of number fields, of Galois group G . Let \mathcal{K} and be the cyclotomic \mathbb{Z}_p -extensions of K , for some prime p . Then, the extension \mathcal{K}/k is Galois. Let $\mathcal{G} := \text{Gal}(\mathcal{K}/k)$. Now, let us assume that the sets \mathcal{S} and \mathcal{T} in \mathcal{K} considered above are \mathcal{G} -invariant. Then, just as above, $\text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p) \otimes \mathbb{Z}_p$ and $\mathcal{A}_{\mathcal{K}, \mathcal{T}}$ are endowed with obvious canonical $\mathbb{Z}_p[[\mathcal{G}]]$ -module structures and the map $\delta \otimes 1$ above is $\mathbb{Z}_p[[\mathcal{G}]]$ -linear. Consequently, $\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}}[p^a]$, for $a \geq 1$, and $T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}})$ are endowed with a canonical $\mathbb{Z}_p[[\mathcal{G}]]$ -module structure and the sequence in Lemma 3.6 is exact in the category of $\mathbb{Z}_p[[\mathcal{G}]]$ -modules.*

If, in addition, K is a CM-number field (which makes \mathcal{K} of CM-type), \mathcal{S} and \mathcal{T} are j -invariant and p is odd, then $\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}}[p^a]^{\pm}$, for $a \geq 1$, and $T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}})^{\pm}$ have natural $\mathbb{Z}_p[[\mathcal{G}]]$ -module structures. The exact sequence of $\mathbb{Z}_p[[\mathcal{G}]]$ -modules in Lemma 3.6 can be split into a direct sum of two exact sequences of $\mathbb{Z}_p[[\mathcal{G}]]$ -modules

$$(13) \quad 0 \longrightarrow T_p(\mathcal{A}_{\mathcal{K},\mathcal{T}})^{\pm} \longrightarrow T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}})^{\pm} \longrightarrow \text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p)^{\pm} \otimes \mathbb{Z}_p \longrightarrow 0.$$

Remark 3.8. For every $n \geq 0$, let $M_{S_n, T_n}^{K_n} := [\text{Div}_{K_n}(S_n \setminus S_{p,n}) \xrightarrow{\delta_n} A_{K_n, T_n}]$, where δ_n is the divisor-class map defined in the proof of the above Lemma. Of course, $M_{S_n, T_n}^{K_n}$ is not an abstract 1-motive, unless the finite group A_{K_n, T_n} happens to be trivial. Nevertheless, one can define the $\mathbb{Z}_p[\Gamma/\Gamma_n]$ -modules

$$M_{S_n, T_n}^{K_n}[m] := (A_{K_n, T_n} \times_{A_{K_n, T_n}}^m \text{Div}_{K_n}(S_n \setminus S_{p,n})) \otimes \mathbb{Z}/m\mathbb{Z},$$

for every m which is a power of p . For all m and n as above, we have obvious and canonical Λ -module morphisms

$$M_{S_n, T_n}^{K_n}[m] \rightarrow M_{S_{n+1}, T_{n+1}}^{K_{n+1}}[m] \rightarrow \mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}}[m].$$

It is easily proved that these lead to a Λ -module isomorphism

$$\varinjlim_n M_{S_n, T_n}^{K_n}[m] \simeq \mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}}[m].$$

3.3. Linking $T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}})^-$ to the classical Iwasawa modules. In this section, we assume that K/k is a Galois extension of number fields, where K is CM and k is totally real. We fix a prime $p > 2$ and let \mathcal{K} denote the cyclotomic \mathbb{Z}_p -extension of K . We let $G := \text{Gal}(K/k)$, $\mathcal{G} := \text{Gal}(\mathcal{K}/k)$, $\Gamma := \text{Gal}(\mathcal{K}/K)$ and $\Lambda := \mathbb{Z}_p[[\Gamma]]$.

Let \mathcal{S} be a \mathcal{G} -equivariant finite set of finite primes in \mathcal{K} . As in classical Iwasawa theory, we denote by $\mathfrak{X}_{\mathcal{S}}$ the Galois group of the maximal abelian pro- p extension of \mathcal{K} which is unramified away from $\mathcal{S} \cup \mathcal{S}_p$. Then, $\mathfrak{X}_{\mathcal{S}}$ is endowed with the usual canonical $\mathbb{Z}_p[[\mathcal{G}]]$ -module structure (with the \mathcal{G} -action on $\mathfrak{X}_{\mathcal{S}}$ given by lift-and-conjugation.) In particular, $\mathfrak{X}_{\mathcal{S}}$ has a canonical Λ -module structure. Recall Iwasawa's classical theorems stating that $\mathfrak{X}_{\mathcal{S}}$ is a finitely generated Λ -module of rank $r_2(K)$ (the number of complex infinite primes in K) and that $\mathfrak{X}_{\mathcal{S}}^+$ is Λ -torsion and contains no non-trivial finite Λ -submodules (see [14].)

Further, let us assume that $\mu_p \subseteq K$ (i.e. $\mu_{p^\infty} \subseteq \mathcal{K}$.) In what follows, we adopt the notations and definitions of §7.2 in the Appendix. In particular, note that under our current assumptions the Teichmüller component ω_p of the p -adic cyclotomic character $c_p : \mathcal{G} \rightarrow \mathbb{Z}_p^\times$ factors through G .

By the definitions of $\mathfrak{X}_{\mathcal{S}}$ and $\mathcal{K}_{\mathcal{S}, \emptyset}$, Kummer theory leads to perfect \mathbb{Z}_p -bilinear pairings of $\mathbb{Z}_p[[\mathcal{G}]]$ -modules

$$\langle \cdot, \cdot \rangle_m : \mathfrak{X}_{\mathcal{S}}/p^m \mathfrak{X}_{\mathcal{S}} \times \mathcal{K}_{\mathcal{S}, \emptyset}^{(p^m)}/\mathcal{K}^{\times p^m} \longrightarrow \mu_{p^m},$$

for all $m \in \mathbb{Z}_{\geq 1}$, with the \mathcal{G} -equivariance property

$$(14) \quad \langle {}^g x, {}^g y \rangle_m = g(\langle x, y \rangle_m),$$

for all $x \in \mathfrak{X}_{\mathcal{S}}/p^m \mathfrak{X}_{\mathcal{S}}$, $y \in \mathcal{K}_{\mathcal{S}, \emptyset}^{(p^m)}/\mathcal{K}^{\times p^m}$ and $g \in \mathcal{G}$. Consequently, for all m as above, we obtain perfect \mathbb{Z}_p -bilinear pairings

$$\mathfrak{X}_{\mathcal{S}}^+/p^m \mathfrak{X}_{\mathcal{S}}^+ \times (\mathcal{K}_{\mathcal{S}, \emptyset}^{(p^m)}/\mathcal{K}^{\times p^m})^- \longrightarrow \mu_{p^m},$$

with property (14). Now, we use Corollary 3.4 and pass to a projective limit with respect to m and the obvious transition maps in the pairings above to obtain a perfect \mathbb{Z}_p -bilinear, continuous pairing of $\mathbb{Z}_p[[\mathcal{G}]]$ -modules

$$\langle \cdot, \cdot \rangle : \mathfrak{X}_S^+ \times T_p(\mathcal{M}_{S,\emptyset}^\mathcal{K})^- \longrightarrow \mathbb{Z}_p(1),$$

with property (14). Consequently, the following holds.

Lemma 3.9. *The last pairing induces an isomorphism of $\mathbb{Z}_p[[\mathcal{G}]]$ -modules*

$$T_p(\mathcal{M}_{S,\emptyset}^\mathcal{K})^- \simeq \text{Hom}_{\mathbb{Z}_p}(\mathfrak{X}_S^+, \mathbb{Z}_p(1)),$$

where the right-hand side has the $\mathbb{Z}_p[[\mathcal{G}]]$ -module structure given by

$$(\lambda * f)(x) := f((\iota \circ t_1)(\lambda) \cdot x),$$

for all $f \in \text{Hom}_{\mathbb{Z}_p}(\mathfrak{X}_S^+, \mathbb{Z}_p(1))$, $x \in \mathfrak{X}_S^+$ and $\lambda \in \mathbb{Z}_p[[\mathcal{G}]]$.

Proof. For all $x \in \mathfrak{X}_S^+$, $y \in T_p(\mathcal{M}_{S,\emptyset}^\mathcal{K})^-$ and $g \in \mathcal{G}$, we have

$$\langle {}^g x, y \rangle = g(\langle x, {}^{g^{-1}} y \rangle) = c_p(g) \langle x, {}^{g^{-1}} y \rangle = \langle x, (\iota \circ t_1)(g) \cdot y \rangle.$$

This shows that the isomorphism above is at least $\mathbb{Z}_p[\mathcal{G}]$ -linear. However, since it is continuous, it has to be $\mathbb{Z}_p[[\mathcal{G}]]$ -linear, as desired. \square

Remark 3.10. *With notations as above, let \mathcal{T} be a finite, nonempty, \mathcal{G} -invariant set of finite primes in \mathcal{K} , which is disjoint from $\mathcal{S} \cup \mathcal{S}_p$. Then, the group morphism $\mathcal{A}_{\mathcal{K},\mathcal{T}} \twoheadrightarrow \mathcal{A}_{\mathcal{K}}$ induces an obvious morphism of abstract 1-motives $\mathcal{M}_{\mathcal{S},\mathcal{T}}^\mathcal{K} \longrightarrow \mathcal{M}_{\mathcal{S},\emptyset}^\mathcal{K}$. This leads to a morphism $T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^\mathcal{K}) \longrightarrow T_p(\mathcal{M}_{\mathcal{S},\emptyset}^\mathcal{K})$ of $\mathbb{Z}_p[[\mathcal{G}]]$ -modules. As a consequence of (11) this morphism leads to an exact sequence of $\mathbb{Z}_p[[\mathcal{G}]]$ -modules*

$$(15) \quad 0 \longrightarrow T_p(\Delta_{\mathcal{K},\mathcal{T}})^- / \mathbb{Z}_p(1) \longrightarrow T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^\mathcal{K})^- \longrightarrow T_p(\mathcal{M}_{\mathcal{S},\emptyset}^\mathcal{K})^- \longrightarrow 0$$

Therefore, Lemma 3.9 above gives surjective morphisms of $\mathbb{Z}_p[[\mathcal{G}]]$ -modules

$$T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^\mathcal{K})^-(n-1) \twoheadrightarrow T_p(\mathcal{M}_{\mathcal{S},\emptyset}^\mathcal{K})^-(n-1) \simeq (\mathfrak{X}_S^+)^*(n), \quad \forall n \in \mathbb{Z}.$$

Above, $(\mathfrak{X}_S^+)^* := \text{Hom}_{\mathbb{Z}_p}(\mathfrak{X}_S^+, \mathbb{Z}_p)$, endowed with the contravariant \mathcal{G} -action given by ${}^g f(x) := f(g^{-1} \cdot x)$, for all $f \in (\mathfrak{X}_S^+)^*$, $g \in \mathcal{G}$ and $x \in \mathfrak{X}_S^+$.

4. COHOMOLOGICAL TRIVIALITY

Throughout this section, \mathcal{K}/\mathcal{K}' will denote a Galois extension of \mathbb{Z}_p -fields, of Galois group G . We fix two finite sets \mathcal{S} and \mathcal{T} of finite primes in \mathcal{K} , such that $\mathcal{T} \cap (\mathcal{S}_p \cup \mathcal{S}) = \emptyset$. From now on, we assume that $\mathcal{T} \neq \emptyset$ and \mathcal{S} contains the finite ramification locus $\mathcal{S}_{\text{ram}}^{\text{fin}}(\mathcal{K}/\mathcal{K}')$ of \mathcal{K}/\mathcal{K}' . Also, we assume that \mathcal{S} and \mathcal{T} are G -invariant and let \mathcal{S}' and \mathcal{T}' denote the sets consisting of all primes on \mathcal{K}' sitting below primes in \mathcal{S} and \mathcal{T} , respectively.

If \mathcal{K} is of CM-type, j will denote, as usual, the complex conjugation automorphism of \mathcal{K} . Since j commutes with any element in G (as automorphisms of \mathcal{K}), it is easy to check that, in this case, \mathcal{K}' is either totally real or of CM-type, depending on whether $\mathcal{K}' \subseteq \mathcal{K}^+ := \mathcal{K}^{j=1}$ or not. In the case where both \mathcal{K} and \mathcal{K}' are of CM-type, then the complex conjugation automorphism of \mathcal{K}' is the restriction of j to \mathcal{K}' and will be denoted by j as well. If \mathcal{K} is of CM-type, we will assume further that \mathcal{S} and \mathcal{T} are j -invariant as well.

Throughout this section, we assume the vanishing of the Iwasawa μ -invariant of all \mathbb{Z}_p -fields involved.

For simplicity, we let $\mathcal{M} := \mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}}$ and $\mathcal{M}' := \mathcal{M}_{\mathcal{S}', \mathcal{T}'}^{\mathcal{K}'}$. The natural abstract 1-motive morphism $\mathcal{M}' \rightarrow \mathcal{M}$ (see Remark 3.5) induces \mathbb{Z}_p -module morphisms $\mathcal{M}'[p^n] \rightarrow \mathcal{M}[p^n]^G$ and $T_p(\mathcal{M}') \rightarrow T_p(\mathcal{M})^G$. The main goal of this section is to use these morphisms in order to study the $\mathbb{Z}_p[G]$ -module structure of $T_p(\mathcal{M})^-$, in the case where \mathcal{K} is of CM-type and p is odd.

Proposition 4.1. *Assume that m is a non-trivial power of p . Then,*

- (1) *The inclusion $\mathcal{K}'^\times \subseteq \mathcal{K}^\times$ induces a group isomorphism*

$$\mathcal{K}'_{\mathcal{S}', \mathcal{T}'}{}^{(m)} / \mathcal{K}'_{\mathcal{T}'}{}^{\times m} \simeq (\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(m)} / \mathcal{K}_{\mathcal{T}}^{\times m})^G.$$

- (2) *If \mathcal{K} and \mathcal{K}' are of CM-type and p is odd, then the morphism $\mathcal{M}' \rightarrow \mathcal{M}$ of abstract 1-motives induces canonical \mathbb{Z}_p -module isomorphisms*

$$\mathcal{M}'[m]^- \simeq (\mathcal{M}[m]^-)^G, \quad T_p(\mathcal{M}')^- \simeq (T_p(\mathcal{M})^-)^G.$$

Proof. (1) Let us fix an m as above. Since \mathcal{K}/\mathcal{K}' is unramified at finite primes outside of \mathcal{S}' , we have inclusions

$$\mathcal{K}'_{\mathcal{T}'}{}^\times = (\mathcal{K}_{\mathcal{T}}^\times)^G \subseteq \mathcal{K}_{\mathcal{T}}^\times, \quad \mathcal{K}'_{\mathcal{S}', \mathcal{T}'}{}^{(m)} \subseteq (\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(m)})^G \subseteq \mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(m)}.$$

Since \mathcal{T} is non-empty and disjoint from \mathcal{S}_p , the group $\mathcal{K}_{\mathcal{T}}^\times$ has no m -torsion. Therefore, the m -power-map induces a G -invariant group isomorphism $\mathcal{K}_{\mathcal{T}}^\times \simeq \mathcal{K}_{\mathcal{T}}^{\times m}$. If one takes G -invariants in this isomorphism, one obtains

$$\mathcal{K}'_{\mathcal{T}'}{}^{\times m} = (\mathcal{K}_{\mathcal{T}}^{\times m})^G = \mathcal{K}_{\mathcal{T}}^{\times m} \cap \mathcal{K}'^\times.$$

When combined with the displayed inclusions above, this leads to an injection

$$\mathcal{K}'_{\mathcal{S}', \mathcal{T}'}{}^{(m)} / \mathcal{K}'_{\mathcal{T}'}{}^{\times m} \hookrightarrow (\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(m)} / \mathcal{K}_{\mathcal{T}}^{\times m})^G.$$

In order to complete the proof of (1), we need to show that this injection is an isomorphism. For this purpose, we write the first four terms in the long G -cohomology sequence associated to the short exact sequence of $\mathbb{Z}[G]$ -modules

$$1 \longrightarrow \mathcal{K}_{\mathcal{T}}^{\times m} \longrightarrow \mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(m)} \longrightarrow \mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(m)} / \mathcal{K}_{\mathcal{T}}^{\times m} \longrightarrow 1.$$

We obtain an exact sequence of multiplicative groups

$$1 \longrightarrow \mathcal{K}'_{\mathcal{T}'}{}^{\times m} \longrightarrow (\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(m)})^G \longrightarrow (\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(m)} / \mathcal{K}_{\mathcal{T}}^{\times m})^G \longrightarrow H^1(G, \mathcal{K}_{\mathcal{T}}^{\times m}) \longrightarrow \dots$$

Therefore, (1) would be a consequence of the following equalities.

$$(16) \quad H^1(G, \mathcal{K}_{\mathcal{T}}^{\times m}) = 0, \quad (\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(m)})^G = \mathcal{K}'_{\mathcal{S}', \mathcal{T}'}{}^{(m)}.$$

Since $\mathcal{K}_{\mathcal{T}}^\times \simeq \mathcal{K}_{\mathcal{T}}^{\times m}$, the first equality above is equivalent to $H^1(G, \mathcal{K}_{\mathcal{T}}^\times) = 0$. This is proved as follows. We let $\mathcal{K}_{(\mathcal{T})}^\times := \{x \in \mathcal{K}^\times \mid \text{ord}_w(x) = 0, \forall w \in \mathcal{T}\}$. We have short exact sequences of $\mathbb{Z}[G]$ -modules

$$\begin{aligned} 1 \longrightarrow \mathcal{K}_{\mathcal{T}}^\times &\longrightarrow \mathcal{K}_{(\mathcal{T})}^\times \xrightarrow{\text{res}_{\mathcal{T}}} \Delta_{\mathcal{K}, \mathcal{T}} \longrightarrow 1, \\ 1 \longrightarrow \mathcal{K}_{(\mathcal{T})}^\times &\longrightarrow \mathcal{K}^\times \xrightarrow{\text{div}_{\mathcal{K}, \mathcal{T}}} \text{Div}_{\mathcal{K}}(\mathcal{T}) \longrightarrow 0. \end{aligned}$$

Here, we have $\text{res}_{\mathcal{T}}(x) := (x \bmod w)_{w \in \mathcal{T}}$, for all $x \in \mathcal{K}_{(\mathcal{T})}^{\times}$, and $\text{div}_{\mathcal{K}, \mathcal{T}}(x) := \sum_{w \in \mathcal{T}} \text{ord}_w(x) \cdot w$, for all $x \in \mathcal{K}^{\times}$. The maps $\text{res}_{\mathcal{T}}$ and $\text{div}_{\mathcal{K}, \mathcal{T}}$ are surjective as a consequence of the weak approximation theorem applied to the independent valuations of \mathcal{K} corresponding to the primes in \mathcal{T} . Since the extension \mathcal{K}/\mathcal{K}' is unramified at primes in \mathcal{T}' and $\mathcal{T} \cap \mathcal{S}_p = \emptyset$, we have $\mathbb{Z}[G]$ -module isomorphisms

$$(17) \quad \Delta_{\mathcal{K}, \mathcal{T}} \simeq \bigoplus_{w' \in \mathcal{T}'} (\kappa(w)^{\times} \otimes_{\mathbb{Z}[G_w]} \mathbb{Z}[G]), \quad \text{Div}_{\mathcal{K}}(\mathcal{T}) \simeq \bigoplus_{w' \in \mathcal{T}'} (\mathbb{Z}w \otimes_{\mathbb{Z}[G_w]} \mathbb{Z}[G]),$$

where w is a prime in \mathcal{T} sitting above w' and G_w is the decomposition group of w in \mathcal{K}/\mathcal{K}' . Since $G_w \simeq G(\kappa(w)/\kappa(w'))$ and $\kappa(w')$ is a \mathbb{Z}_p -extension of a finite field, G_w is cyclic of order coprime to p . The first isomorphism in (17) combined with Shapiro's Lemma, Hilbert's Theorem 90 and Herbrandt quotient theory (view $\kappa(w)$ as a union of Galois extensions of Galois group G_w of finite subfields of $\kappa(w')$) give

$$\hat{H}^i(G, \Delta_{\mathcal{K}, \mathcal{T}}) \simeq \bigoplus_{w' \in \mathcal{T}'} \hat{H}^i(G_w, \kappa(w)^{\times}) = 0, \quad \forall i \in \mathbb{Z}.$$

Consequently, Tate cohomology applied to the first exact sequences above gives

$$H^1(G, \mathcal{K}_{\mathcal{T}}^{\times}) \simeq H^1(G, \mathcal{K}_{(\mathcal{T})}^{\times}).$$

The second isomorphism in (17) combined with Shapiro's Lemma gives

$$\text{Div}_{\mathcal{K}'}(\mathcal{T}') \simeq \text{Div}_{\mathcal{K}}(\mathcal{T})^G, \quad H^1(G, \text{Div}_{\mathcal{K}}(\mathcal{T})) \simeq \bigoplus_{w' \in \mathcal{T}'} H^1(G_w, \mathbb{Z}) = 0,$$

where the first isomorphism above is induced by the canonical injection $\text{Div}_{\mathcal{K}'}(\mathcal{T}') \rightarrow \text{Div}_{\mathcal{K}}(\mathcal{T})$. Since $\text{div}_{\mathcal{K}, \mathcal{T}}|_{\mathcal{K}'^{\times}} = \text{div}_{\mathcal{K}, \mathcal{T}}$, Tate cohomology applied to the second exact sequence above combined with Hilbert's Theorem 90 gives

$$H^1(G, \mathcal{K}_{(\mathcal{T})}^{\times}) \simeq H^1(G, \mathcal{K}^{\times}) = 0.$$

Consequently, we have

$$H^1(G, \mathcal{K}_{\mathcal{T}}^{\times}) \simeq H^1(G, \mathcal{K}_{(\mathcal{T})}^{\times}) \simeq H^1(G, \mathcal{K}^{\times}) = 0,$$

which concludes the proof of the first equality in (16).

In order to prove the second equality in (16), we consider the exact sequence

$$1 \longrightarrow \mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(m)} \longrightarrow \mathcal{K}_{\mathcal{T}}^{\times} \xrightarrow{\overline{\text{div}}_{\mathcal{K}, \mathcal{S}}} \text{Div}_{\mathcal{K}, \mathcal{S}} \otimes \mathbb{Z}/m\mathbb{Z}$$

in the category of $\mathbb{Z}[G]$ -modules, where $\text{Div}_{\mathcal{K}, \mathcal{S}} := \bigoplus_{v \notin \mathcal{S}} \Gamma_v \cdot v$ is the group of divisors in \mathcal{K} supported away from \mathcal{S} and $\overline{\text{div}}_{\mathcal{K}, \mathcal{S}}(x) := (\sum_{v \notin \mathcal{S}} \text{ord}_v(x) \cdot v) \otimes \hat{1}$, for all $x \in \mathcal{K}_{\mathcal{T}}^{\times}$. Since \mathcal{K}/\mathcal{K}' is unramified away from \mathcal{S} , the natural injective morphism $\text{Div}_{\mathcal{K}', \mathcal{S}'} \rightarrow \text{Div}_{\mathcal{K}, \mathcal{S}}$ induces an isomorphism of groups $\text{Div}_{\mathcal{K}', \mathcal{S}'} \simeq (\text{Div}_{\mathcal{K}, \mathcal{S}})^G$ and the restriction of $\overline{\text{div}}_{\mathcal{K}, \mathcal{S}}$ to $\mathcal{K}'_{\mathcal{T}'}^{\times}$ equals $\overline{\text{div}}_{\mathcal{K}', \mathcal{S}'}$. Consequently, when we take G -invariants in the exact sequence above, we obtain an exact sequence at the \mathcal{K}' -level

$$1 \longrightarrow (\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(m)})^G \longrightarrow \mathcal{K}'_{\mathcal{T}'}^{\times} \xrightarrow{\overline{\text{div}}_{\mathcal{K}', \mathcal{S}'}} \text{Div}_{\mathcal{K}', \mathcal{S}'} \otimes \mathbb{Z}/m\mathbb{Z},$$

which shows that $\mathcal{K}'_{\mathcal{S}', \mathcal{T}'} = (\mathcal{K}_{\mathcal{S}, \mathcal{T}}^{(m)})^G$, as desired.

(2) The first isomorphism in part (2) of the Proposition is a direct consequence of part (1) combined with Remark 3.5 and Corollary 3.4. The second isomorphism in part (2) is obtained from the first by taking the obvious projective limit. \square

Let us assume that \mathcal{K} is of CM-type and p is odd. Then, if we let \mathbb{F}_p denote the field with p elements, we have an exact sequence of $\mathbb{F}_p[G]$ -modules

$$1 \longrightarrow \mathcal{A}_{\mathcal{K},\mathcal{T}}^-[p] \longrightarrow \mathcal{M}[p]^- \longrightarrow (\text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p) \otimes \mathbb{Z}/p\mathbb{Z})^- \longrightarrow 1.$$

According to Lemma 2.8, if we let $r_{\mathcal{M}}^- := \dim_{\mathbb{F}_p} \mathcal{M}[p]^-$, we have

$$(18) \quad r_{\mathcal{M}}^- = \lambda_{\mathcal{K}}^- + \delta_{\mathcal{K},\mathcal{T}}^- - \delta_{\mathcal{K}} + d_{\mathcal{K},\mathcal{S}}^-,$$

where $\lambda_{\mathcal{K}}^-$, $\delta_{\mathcal{K},\mathcal{T}}^-$ and $\delta_{\mathcal{K}}$ are as in Lemma 2.8 and

$$d_{\mathcal{K},\mathcal{S}}^- := \dim_{\mathbb{F}_p} (\text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p) \otimes \mathbb{Z}/p\mathbb{Z})^-.$$

We need a concrete formula for $d_{\mathcal{K},\mathcal{S}}^-$. For that purpose, we let $J := G(\mathcal{K}/\mathcal{K}^+)$ and we view $\text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p)$ as a $\mathbb{Z}[J]$ -module. Obviously, J is cyclic of order 2, generated by j . Let \mathcal{S}^+ and \mathcal{S}_p^+ denote the sets of primes in \mathcal{K}^+ sitting below primes in \mathcal{S} and \mathcal{S}_p , respectively. Also, for every $v \in \mathcal{S}^+ \setminus \mathcal{S}_p^+$, let J_v be the decomposition group of v in $\mathcal{K}/\mathcal{K}^+$. Then, we have the following obvious $\mathbb{Z}[J]$ -module and \mathbb{F}_p -vector space isomorphisms, respectively:

$$\text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p) \simeq \bigoplus_{v \in \mathcal{S}^+ \setminus \mathcal{S}_p^+} \mathbb{Z}[J/J_v], \quad (\text{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p) \otimes \mathbb{Z}/p\mathbb{Z})^- \simeq \bigoplus_{v \in \mathcal{S}^+ \setminus \mathcal{S}_p^+} \mathbb{F}_p[J/J_v]^-.$$

Obviously, we have $\mathbb{F}_p[J/J_v]^- \simeq \mathbb{F}_p$, if J_v is trivial (i.e. if v splits in $\mathcal{K}/\mathcal{K}^+$) and $\mathbb{F}_p[J/J_v]^- = 0$, otherwise. Consequently, we have the following formula for $d_{\mathcal{K},\mathcal{S}}^-$:

$$d_{\mathcal{K},\mathcal{S}}^- = \text{card} \{v \mid v \in \mathcal{S}^+ \setminus \mathcal{S}_p^+, v \text{ splits completely in } \mathcal{K}/\mathcal{K}^+\}.$$

The above formula permits us to reformulate the main result in [16] as follows.

Theorem 4.2 (Kida). *Assume that p is odd, G is a p -group and \mathcal{K} and \mathcal{K}' are of CM-type. Then, we have the following equality.*

$$(\lambda_{\mathcal{K}}^- - \delta_{\mathcal{K}} + d_{\mathcal{K},\mathcal{S}}^-) = |G| \cdot (\lambda_{\mathcal{K}'}^- - \delta_{\mathcal{K}'} + d_{\mathcal{K}',\mathcal{S}'}^-).$$

The above result is known as “Kida’s formula”. It was first proved by Kida in [16] and later and with different methods by Iwasawa (see [15]) and Sinnott (see [29]). This should be viewed as a “partial” number field analogue of Hurwitz’s genus formula for finite covers of smooth, projective curves over an algebraically closed field. Here, $\mathcal{A}_{\mathcal{K}}^-$ plays the role of the Jacobian and its corank (equal to $\lambda_{\mathcal{K}}^-$) plays the role of twice the genus of the underlying curve. Of course, a “full” number field analogue of the Hurwitz genus formula should involve the ideal class-group $\mathcal{A}_{\mathcal{K}}$ and its corank $\lambda_{\mathcal{K}}$. To our knowledge, no such formula exists at the moment. The following consequence of the above theorem is of interest to us.

Corollary 4.3. *Assume that p is odd, G is a p -group and \mathcal{K} and \mathcal{K}' are of CM-type. Then, we have the following equality.*

$$r_{\mathcal{M}}^- = |G| \cdot r_{\mathcal{M}'}^-.$$

Proof. The first isomorphism in (17) induces an isomorphism of $\mathbb{Z}_p[G]$ -modules

$$(\Delta_{\mathcal{K}, \mathcal{T}} \otimes \mathbb{Z}_p)^- \simeq (\Delta_{\mathcal{K}', \mathcal{T}} \otimes \mathbb{Z}_p)^- \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G].$$

Above, we took into account that since G is a p -group, G_w is trivial, for all $w \in \mathcal{T}$. Now, by comparing p -coranks on both sides of the above isomorphism, we get

$$\delta_{\mathcal{K}, \mathcal{T}}^- = |G| \cdot \delta_{\mathcal{K}', \mathcal{T}'}^-.$$

The equality in the statement of the corollary is a direct consequence of the above equality combined with (18) and Theorem 4.2. \square

Now, we are ready to state and prove the main results of this section. The reader will note that in what follows we are using a strategy similar to that of §3 in [13].

Theorem 4.4. *Assume that G is a p -group, \mathcal{K} and \mathcal{K}' are of CM-type, and p is odd. Then, the following hold.*

- (1) *The $\mathbb{F}_p[G]$ -module $\mathcal{M}[p]^-$ is free of rank*

$$r_{\mathcal{M}'}^- := \lambda_{\mathcal{K}'}^- + \delta_{\mathcal{K}', \mathcal{T}'}^- + d_{\mathcal{K}', \mathcal{S}'}^- - \delta_{\mathcal{K}'}^-.$$

- (2) *The $\mathbb{Z}_p[G]$ -module $T_p(\mathcal{M})^-$ is free of rank $r_{\mathcal{M}'}^-$.*

Proof. For the proof of (1), we need the following (see [20], Proposition 2, §4.)

Lemma 4.5 (Nakajima). *Assume that k is a field of characteristic $\ell \neq 0$, G is a finite ℓ -group and M is a finitely generated $k[G]$ -module, such that*

$$\dim_k M \geq |G| \cdot \dim_k M^G.$$

Then, M is a free $k[G]$ -module of rank equal to $\dim_k M^G$.

Now, if we combine the first isomorphism in Proposition 4.1(2) for $m := p$, with Corollary 4.3 above, we obtain the following.

$$\dim_{\mathbb{F}_p} \mathcal{M}[p]^- = |G| \cdot \dim_{\mathbb{F}_p} (\mathcal{M}[p]^-)^G.$$

The above Lemma applied to $\ell := p$, $k := \mathbb{F}_p$ and $M := \mathcal{M}[p]^-$ implies that, indeed, $\mathcal{M}[p]^-$ is a free $\mathbb{F}_p[G]$ -module whose rank satisfies

$$\text{rank}_{\mathbb{F}_p[G]} \mathcal{M}[p]^- = \dim_{\mathbb{F}_p} (\mathcal{M}[p]^-)^G = \dim_{\mathbb{F}_p} \mathcal{M}'[p]^- = r_{\mathcal{M}'}^-.$$

This concludes the proof of part (1).

In order to prove part (2), we consider the exact sequence of $\mathbb{Z}_p[G]$ -modules

$$0 \longrightarrow T_p(\mathcal{M})^- \xrightarrow{\times p} T_p(\mathcal{M})^- \longrightarrow \mathcal{M}[p]^- \longrightarrow 0$$

where the injective morphism is the multiplication-by- p map (see (7).) Since $\mathcal{M}[p]^-$ is a free $\mathbb{F}_p[G]$ -module, it is G -induced and therefore G -cohomologically trivial. Consequently, when we apply Tate G -cohomology to the short exact sequence above, we obtain group-isomorphisms

$$\widehat{H}^i(G, T_p(\mathcal{M})^-) \xrightarrow{\times p} \widehat{H}^i(G, T_p(\mathcal{M})^-)$$

given by the multiplication-by- p map, for all $i \in \mathbb{Z}$. Since the cohomology groups above are p -groups, this implies that $\widehat{H}^i(G, T_p(\mathcal{M})^-) = 0$. Since G is a p -group,

this implies that $T_p(\mathcal{M})^-$ is cohomologically trivial (see Theorem 8, in [27], Ch. IX, §5.) Therefore, $T_p(\mathcal{M})^-$ is a projective $\mathbb{Z}_p[G]$ -module, as it is \mathbb{Z}_p -free (see loc.cit.) However, since G is a p -group, the ring $\mathbb{Z}_p[G]$ is local and therefore $T_p(\mathcal{M})^-$ is a free $\mathbb{Z}_p[G]$ -module (see [19], Lemma 1.2.) The $\mathbb{F}_p[G]$ -module isomorphism

$$T_p(\mathcal{M})^- = \mathbb{F}_p[G] \otimes_{\mathbb{Z}_p[G]} T_p(\mathcal{M})^-$$

(a consequence of the short exact sequence above) combined with part (1) leads to

$$\text{rank}_{\mathbb{Z}_p[G]} T_p(\mathcal{M})^- = \text{rank}_{\mathbb{F}_p[G]} \mathcal{M}[p]^- = r_{\mathcal{M}}^-.$$

This concludes the proof of the Theorem. \square

Theorem 4.6. *Assume that \mathcal{K} is of CM-type, p is odd and G is arbitrary. Then, if we let “pd” denote “projective dimension”, the following hold.*

- (1) $\text{pd}_{\mathbb{Z}_p[G]} T_p(\mathcal{M})^- = \text{pd}_{\mathbb{Z}_p} T_p(\mathcal{M})^- = 0$.
- (2) *If, in addition, \mathcal{K}' is the \mathbb{Z}_p -cyclotomic extension of a number field k , such that \mathcal{K}/k is Galois of Galois group \mathcal{G} and \mathcal{S} and \mathcal{T} are \mathcal{G} -invariant, then $\text{pd}_{\mathbb{Z}_p[[\mathcal{G}]]} T_p(\mathcal{M})^- = \text{pd}_{\mathbb{Z}_p[[\mathcal{G}]]} T_p(\mathcal{M})^- = 1$, unless $T_p(\mathcal{M})^- = 0$.*

Proof. (1) Since $T_p(\mathcal{M})^-$ is a free \mathbb{Z}_p -module, the statement in part (1) of the Theorem is equivalent to the G -cohomological triviality of $T_p(\mathcal{M})^-$. (See [27], Ch. IX, §5.) In order to prove that, let H denote a p -Sylow subgroup of G and let \mathcal{K}^H be the maximal subfield of \mathcal{K} fixed by H . Then, $\mathcal{K}/\mathcal{K}^H$ is Galois, of Galois group H . Also, \mathcal{K}^H is of CM-type. Indeed, if $\mathcal{K}^H \subseteq \mathcal{K}^+$, then $2 = [\mathcal{K} : \mathcal{K}^+]$ would divide $|H| = [\mathcal{K} : \mathcal{K}^H]$, which is false, because p is odd. Consequently, we can apply Theorem 4.4(2) to the extension $\mathcal{K}/\mathcal{K}^H$ to conclude that

$$\widehat{H}^i(H, T_p(\mathcal{M})^-) = 0, \quad \forall i \in \mathbb{Z}.$$

Since this happens for any p -Sylow subgroup H of G , the module $T_p(\mathcal{M})^-$ is G -cohomologically trivial and therefore $\mathbb{Z}_p[G]$ -projective, as desired. (See loc.cit.)

(2) Let $\Gamma_k := G(\mathcal{K}'/k)$. Since Γ_k is a free abelian topological group, the usual Galois-restriction exact sequence

$$(19) \quad 1 \longrightarrow G \longrightarrow \mathcal{G} \xrightarrow{\text{res}_{\mathcal{K}/\mathcal{K}'}} \Gamma_k \longrightarrow 1$$

is split. Pick a splitting and let Γ be its image in \mathcal{G} . Then, we have a group isomorphism $\mathcal{G} \simeq G \rtimes \Gamma$, where \rtimes denotes a semi-direct product. Let K be the normal closure (over k) in \mathcal{K} of \mathcal{K}^Γ (the maximal subfield of \mathcal{K} fixed by Γ .) Then K/k is a Galois extension of number fields and $\mathcal{K} = K \cdot \mathcal{K}'$ (the compositum of K and \mathcal{K}' inside \mathcal{K} .) Consequently, if we let $G' := G(K/k)$, Galois restriction induces an isomorphism between G and the normal subgroup $G(K/\mathcal{K}' \cap K)$ of G' . Via this isomorphism, we identify G with $G(K/\mathcal{K}' \cap K)$ and G'/G with $G(\mathcal{K}' \cap K/k)$. We have an exact sequence of groups induced by the obvious Galois restriction maps

$$(20) \quad 1 \longrightarrow \mathcal{G} \longrightarrow G' \times \Gamma \longrightarrow G'/G \longrightarrow 1.$$

Consequently, $\mathbb{Z}_p[[G' \times \Gamma]]$ viewed as a left or right $\mathbb{Z}_p[[\mathcal{G}]]$ -module in the obvious way is free of basis given by any complete set of representatives for the left cosets of G in G' . We consider the left $\mathbb{Z}_p[[G' \times \Gamma]]$ -module

$$\widetilde{T_p(\mathcal{M})^-} := \mathbb{Z}_p[[G' \times \Gamma]] \otimes_{\mathbb{Z}_p[[\mathcal{G}]]} T_p(\mathcal{M})^-.$$

As $\mathbb{Z}_p[[G' \times \Gamma]]$ is a free right $\mathbb{Z}_p[[\Gamma]]$ -module, (2) in the Theorem is equivalent to

$$\mathrm{pd}_{\mathbb{Z}_p[[G' \times \Gamma]]} \widetilde{T_p(\mathcal{M})}^- = 1.$$

We claim that this follows directly from part (1) of the Theorem combined with Proposition 2.2 and Lemma 2.3 in [23]. In order to see this, let us first note that we have an obvious ring isomorphism $\mathbb{Z}_p[[G' \times \Gamma]] \simeq \Lambda[G']$, where $\Lambda := \mathbb{Z}_p[[\Gamma]]$. Now, we have the following.

Lemma 4.7. *A finitely generated $\Lambda[G']$ -module M satisfies $\mathrm{pd}_{\Lambda[G']} M \leq 1$ if and only if the following conditions are satisfied.*

- (i) $\mathrm{pd}_{\mathbb{Z}_p[G']} M \leq 1$.
- (ii) M has no non trivial finite Λ -submodules (i.e. $\mathrm{pd}_{\Lambda} M \leq 1$.)

Proof. Combine Proposition 2.2 and Lemma 2.3 in [23]. □

Since the Λ -module $\widetilde{T_p(\mathcal{M})}^-$ is \mathbb{Z}_p -free, it satisfies (ii) above automatically. Also, as a consequence of the definitions and exact sequence (20), we have an isomorphism of left $\mathbb{Z}_p[G']$ -modules

$$\widetilde{T_p(\mathcal{M})}^- \simeq \mathbb{Z}_p[G'] \otimes_{\mathbb{Z}_p[G]} T_p(\mathcal{M})^-.$$

Consequently, part (1) of the Theorem implies that $\mathrm{pd}_{\mathbb{Z}_p[G']} \widetilde{T_p(\mathcal{M})}^- = 0$. The above Lemma implies that $\mathrm{pd}_{\Lambda[G']} \widetilde{T_p(\mathcal{M})}^- \leq 1$. However, since $\widetilde{T_p(\mathcal{M})}^-$ is of finite \mathbb{Z}_p -rank, we must have $\mathrm{pd}_{\Lambda[G']} \widetilde{T_p(\mathcal{M})}^- = 1$. This concludes the proof of part (2) in the Theorem. □

Corollary 4.8. *Assume that \mathcal{K} and \mathcal{K}' are of CM-type, p is odd and G is arbitrary. Then, there is a canonical isomorphism of \mathbb{Z}_p -modules*

$$T_p(\mathcal{M})^- / I_G T_p(\mathcal{M})^- \simeq T_p(\mathcal{M}')^-,$$

where I_G is the usual augmentation ideal in $\mathbb{Z}_p[G]$.

Proof. Since $T_p(\mathcal{M})^-$ is $\mathbb{Z}_p[G]$ -projective, it is G -cohomologically trivial. In particular, $\widehat{H}^0(G, T_p(\mathcal{M})^-) = \widehat{H}^{-1}(G, T_p(\mathcal{M})^-) = 0$. As a consequence, if we denote by N_G the usual norm element in $\mathbb{Z}_p[G]$, we have

$$(T_p(\mathcal{M})^-)^G = N_G \cdot T_p(\mathcal{M})^-, \quad T_p(\mathcal{M})^- / I_G T_p(\mathcal{M})^- \simeq N_G \cdot T_p(\mathcal{M})^-,$$

where the isomorphism of \mathbb{Z}_p -modules to the right is induced by multiplication with N_G . Now, the corollary is a direct consequence of Proposition 4.1, part (2). □

5. THE EQUIVARIANT MAIN CONJECTURE

We begin by recalling two classical theorems on special values of global L -functions. Let K/k be an abelian extension of number fields of Galois group G . Throughout, if \mathcal{C} is an algebraically closed field, we denote by $\widehat{G}(\mathcal{C})$ the group of irreducible \mathcal{C} -valued characters of G . Let S be a finite set of primes in k , which contains the set $S_\infty(k) \cup S_{\mathrm{ram}}(K/k)$ of primes which are either infinite or ramify

in K/k . For every $\chi \in \widehat{G}(\mathbb{C})$, we let $L_S(\chi, s)$ denote the \mathbb{C} -valued S -incomplete Artin L -function associated to χ , of complex variable s . These L -functions are holomorphic everywhere, except for a simple pole at $s = 1$ when $\chi = \mathbf{1}_G$. We let

$$\Theta_{S,K/k} := \sum_{\chi \in \widehat{G}(\mathbb{C})} L_S(\chi^{-1}, s) \cdot e_\chi, \quad \Theta_{S,K/k} : \mathbb{C} \rightarrow \mathbb{C}[G]$$

denote the so-called S -incomplete G -equivariant L -function associated to K/k , where $e_\chi := 1/|G| \sum_{\sigma \in G} \chi(\sigma) \cdot \sigma^{-1}$ is the idempotent associated to χ in $\mathbb{C}[G]$.

Theorem 5.1 (Siegel [28]). *For all $n \in \mathbb{Z}_{\geq 1}$, we have*

$$\Theta_{S,K/k}(1 - m) \in \mathbb{Q}[G].$$

As usual, for every abelian group M and every $m \in \mathbb{Z}$, we let $M(m)$ denote the group M endowed with the action by the absolute Galois group G_k of k given by the n -th power of the cyclotomic character $c_k : G_k \rightarrow \text{Aut}(\mu_\infty) \simeq \widehat{\mathbb{Z}}^\times$.

Theorem 5.2 (Deligne-Ribet [7], Cassou-Noguès [3]). *For all $m \in \mathbb{Z}_{\geq 1}$, we have*

$$\text{Ann}_{\mathbb{Z}[G]}(\mathbb{Q}/\mathbb{Z}(m)^{G_K}) \cdot \Theta_{S,K/k}(1 - m) \subseteq \mathbb{Z}[G],$$

where $\mathbb{Q}/\mathbb{Z}(m)^{G_K}$ denotes the $\mathbb{Z}[G]$ -module of G_K -invariants of $\mathbb{Q}/\mathbb{Z}(m)$.

Now, let T be a finite, non-empty set of finite primes in k , disjoint from S . We let

$$\delta_{T,K/k} := \prod_{v \in T} (1 - \sigma_v^{-1} \cdot (\mathbf{N}v)^{1-s}), \quad \delta_{T,K/k} : \mathbb{C} \rightarrow \mathbb{C}[G],$$

where σ_v denotes the Frobenius morphism associated to v in G .

Corollary 5.3. *Let p be a prime number and $m \in \mathbb{Z}_{\geq 1}$. Assume that either T contains at least a prime which does not sit above p or p does not divide the cardinality of (the finite group) $\mathbb{Q}/\mathbb{Z}(m)^{G_K}$. Then, we have*

$$\delta_{T,K/k}(1 - m) \cdot \Theta_{S,K/k}(1 - m) \in \mathbb{Z}_{(p)}[G].$$

Proof. Remark that if $v \in T$ does not sit above p , then σ_v acts on the group $\mathbb{Q}/\mathbb{Z}(m)^{G_K} \otimes \mathbb{Z}_p \simeq \mathbb{Q}_p/\mathbb{Z}_p(m)^{G_K}$ via multiplication by $(\mathbf{N}v)^m$. This implies that, under the hypotheses of the Corollary, we have

$$\delta_{T,K/k}(1 - m) \in \text{Ann}_{\mathbb{Z}_{(p)}[G]}(\mathbb{Q}_p/\mathbb{Z}_p(m)^{G_K}).$$

Now, the Corollary follows from Theorem 5.2. \square

Definition 5.4. *For S and T as above, the S -incomplete T -modified G -equivariant L -function associated to K/k is defined by*

$$\Theta_{S,T,K/k} := \delta_{T,K/k}(s) \cdot \Theta_{S,K/k}(s), \quad \Theta_{S,T,K/k} : \mathbb{C} \rightarrow \mathbb{C}[G].$$

Remark 5.5. *Note that $\Theta_{S,T,K/k}$ is holomorphic everywhere on \mathbb{C} and*

$$\Theta_{S,T,K/k}(1 - m) \in \mathbb{Z}_{(p)}[G],$$

whenever the integer m , the prime p and the set T satisfy the hypotheses of the Corollary above. In particular, if T contains a prime of residual characteristics coprime to the cardinality of $(\mathbb{Q}/\mathbb{Z}(m)^{G_K})$, for some $m \in \mathbb{Z}_{\geq 1}$, then

$$\Theta_{S,T,K/k}(1 - m) \in \mathbb{Z}[G].$$

Also, if T contains two primes of distinct residual characteristics, then

$$\Theta_{S,T,K/k}(1-m) \in \mathbb{Z}[G], \quad \forall m \in \mathbb{Z}_{\geq 1}.$$

Throughout the rest of this section, we fix an odd prime p and an abelian extension \mathcal{K}/k , where \mathcal{K} is a CM \mathbb{Z}_p -field and k is a totally real number field. Let $\mathcal{G} := \text{Gal}(\mathcal{K}/k)$. We fix two finite, non-empty, disjoint sets S and T of primes in k , such that S contains $S_{\text{ram}}(\mathcal{K}/k) \cup S_{\infty}$. Note that, in particular, S contains the set S_p of p -adic primes of k . We let \mathcal{S} and \mathcal{T} denote the sets of finite primes in \mathcal{K} sitting above primes in S and T , respectively.

The main goal of this section is the proof of an Equivariant Main Conjecture for the data $(\mathcal{K}/k, S, T, p)$. This statement expresses the first Fitting invariant $\text{Fit}_{\mathbb{Z}_p[[\mathcal{G}]]^-}(T_p(\mathcal{M}_{S,T}^{\mathcal{K}})^-)$ of the finitely generated module $T_p(\mathcal{M}_{S,T}^{\mathcal{K}})^-$ over the (Noetherian, commutative) ring

$$\mathbb{Z}_p[[\mathcal{G}]]^- := \mathbb{Z}_p[[\mathcal{G}]]/(1+j)$$

in terms of a certain equivariant p -adic L -function $\Theta_{S,T}^{(\infty)}$, which is an element of $\mathbb{Z}_p[[\mathcal{G}]]^-$ canonically associated to the data $(\mathcal{K}/k, S, T, p)$. As usual, j denotes the complex conjugation automorphism of \mathcal{K} , viewed as an element of \mathcal{G} . The precise statement, to be proved in §5.3 below, is the following.

Theorem 5.6 (The Equivariant Main Conjecture). *Under the above hypotheses, we have the following equality of ideals in $\mathbb{Z}_p[[\mathcal{G}]]^-$.*

$$\text{Fit}_{\mathbb{Z}_p[[\mathcal{G}]]^-}(T_p(\mathcal{M}_{S,T}^{\mathcal{K}})^-) = (\Theta_{S,T}^{(\infty)}).$$

Let \mathcal{K}' be the cyclotomic \mathbb{Z}_p -extension of k and $\Gamma_k := G(\mathcal{K}'/k)$. As in the proof of Theorem 4.6(2), the exact sequence

$$1 \longrightarrow G(\mathcal{K}/\mathcal{K}') \longrightarrow \mathcal{G} \xrightarrow{\text{res}_{\mathcal{K}/\mathcal{K}'}} \Gamma_k \longrightarrow 1$$

is split. We fix a splitting, i.e. we fix a subgroup Γ of \mathcal{G} which is mapped isomorphically to Γ_k by $\text{res}_{\mathcal{K}/\mathcal{K}'}$. We let $K := \mathcal{K}^{\Gamma}$ denote the maximal subfield of \mathcal{K} fixed by Γ . Then, Galois theory combined with the fact that \mathcal{G} is abelian, implies that K/k is Galois, $K \cdot \mathcal{K}' = \mathcal{K}$ and $\mathcal{K}' \cap K = k$. If we let $G := G(K/k)$, Galois restriction induces group isomorphisms $\mathcal{G} \simeq G \times \Gamma_k$, $\Gamma \simeq \Gamma_k$, and $G(\mathcal{K}/\mathcal{K}') \simeq G$. Below, we freely identify these groups via these isomorphisms. Obviously, K is a CM number field whose cyclotomic \mathbb{Z}_p -extension is \mathcal{K} . We let j denote the complex conjugation automorphism of K as well. We fix a topological generator γ of Γ and identify it via the Galois restriction isomorphism with a generator of Γ_k . For any finite extension \mathcal{O} of \mathbb{Z}_p , we have isomorphisms of compact $\mathcal{O}[G]$ -algebras

$$(21) \quad \mathcal{O}[[\mathcal{G}]] \simeq \mathcal{O}[G][[\Gamma]] \simeq \mathcal{O}[G][[t]], \quad \mathcal{O}[[\mathcal{G}]]^{\pm} \simeq \mathcal{O}[G]^{\pm}[[\Gamma]] \simeq \mathcal{O}[G]^{\pm}[[t]].$$

Above, $\mathcal{O}[G]^{\pm} := \mathcal{O}[G]/(1 \mp j)$, t is a variable, and the right-most isomorphisms send γ to $(t+1)$, as usual.

5.1. The work of Wiles on the Main Conjecture [34]. In order to simplify notations, in this section we assume that $\mu_p \subseteq K$ (i.e. $\mu_{p^{\infty}} \subseteq \mathcal{K}$.) Let $c := \omega \cdot \kappa$ be the decomposition of the p -cyclotomic character $c := c_p$ of \mathcal{G} into its Teichmüller component $\omega := \omega_p$ and its complement $\kappa := \kappa_p$, as in §7.2 in the Appendix. Note

that, in this case, ω and κ factor through G and Γ , respectively. Let u be the element in $(1 + p\mathbb{Z}_p)$ given by $u := \kappa(\gamma)$.

We fix an embedding $\mathbb{C} \hookrightarrow \mathbb{C}_p$. Via this embedding we identify $\widehat{G}(\mathbb{C})$ and $\widehat{G}(\mathbb{C}_p)$. A character $\psi \in \widehat{G}(\mathbb{C}_p)$ is called even if $\psi(j) = 1$ and odd otherwise. Let \mathcal{O} be a fixed finite extension of \mathbb{Z}_p which contains the values of all $\psi \in \widehat{G}(\mathbb{C}_p)$. We fix a uniformizer π in \mathcal{O} and denote by $Q(\mathcal{O})$ the field of fractions of \mathcal{O} . In [7], Deligne and Ribet proved that for every character $\psi \in \widehat{G}(\mathbb{C}_p)$, there exist power series $G_{\psi,S}(t)$ and $H_{\psi,S}(t)$ in $\mathcal{O}[[t]]$, uniquely determined by the following properties.

$$(22) \quad H_{\psi,S} = \begin{cases} t, & \text{if } \psi = \mathbf{1}_G; \\ 1, & \text{otherwise;} \end{cases} \quad \frac{G_{\psi,S}(u^m - 1)}{H_{\psi,S}(u^m - 1)} = L_S(\psi\omega^{-m}, 1-m), \quad \forall m \in \mathbb{Z}_{\geq 1}.$$

The reader may consult §1 in [34] and §4 in [23] for the properties above and note that, since $K \cap K' = k$, all the non trivial characters of G are of “type S”, in the terminology used in loc.cit.

By definition, the S -incomplete p -adic L -function $L_{p,S}(\psi, s)$ associated to an even character $\psi \in \widehat{G}(\mathbb{C})$ is given by

$$(23) \quad L_{p,S}(\psi, 1-s) = \frac{G_{\psi,S}(u^s - 1)}{H_{\psi,S}(u^s - 1)}, \quad s \in \mathbb{Z}_p.$$

The function $L_{p,S}(\psi, s)$ is p -adically analytic everywhere, except for a possible pole of order 1 at $s = 1$, if $\psi = \mathbf{1}_G$.

Remark 5.7. *As an immediate consequence of the functional equation for the global L -functions $L_S(\chi, s)$ considered above, we have $L_S(\chi, 1-m) = 0$, whenever χ and $1-m$ have the same parity, for all $m \in \mathbb{Z}_{\geq 1}$ and all $\chi \in \widehat{G}(\mathbb{C})$. Assume that if ψ is an odd character. Then $\psi\omega^{-m}$ and $1-m$ have the same parity, for all $m \in \mathbb{Z}_{\geq 1}$. Consequently, we have $L_S(\psi\omega^{-m}, 1-m) = 0$, for all $m \in \mathbb{Z}_{\geq 1}$. Therefore, $G_{\psi,S}(t) = 0$ and $L_{p,S}(\psi, s) = 0$, for all odd characters ψ .*

For simplicity, if L is a subfield of K containing k , we denote by \mathfrak{X}_L the Galois group of the maximal abelian pro- p extension of $\mathcal{L} := L \cdot K'$ (the cyclotomic \mathbb{Z}_p -extension of L), which is unramified away from the primes above those in S . We view \mathfrak{X}_L as a module over $\mathbb{Z}_p[[G(\mathcal{L}/k)]] \simeq \mathbb{Z}_p[G(L/k)][[\Gamma]]$ (and consequently over $\mathbb{Z}_p[[\mathcal{G}]]$) in the usual way. Note that \mathfrak{X}_K is the module we denoted by \mathfrak{X}_S in §3.3. We will continue to use \mathfrak{X}_S instead of \mathfrak{X}_K in what follows. The modules \mathfrak{X}_L and \mathfrak{X}_L^+ are finitely generated, respectively torsion and finitely generated modules over $\Lambda := \mathbb{Z}_p[[\Gamma]]$ (a classical theorem of Iwasawa [14].)

Let $\psi \in \widehat{G}(\mathbb{C}_p)$ be an even character. Let $K_\psi := K^{\ker \psi}$ be the largest subfield of K fixed by the kernel of ψ . We have $G(K_\psi/k) \simeq F \times H$, where H is the p -Sylow subgroup of $G(K_\psi/k)$ and F is its maximal subgroup of order coprime to p . Consequently, ψ (viewed as a character of $G(K_\psi/k)$) splits as $\psi = \varphi \cdot \rho$, with φ a faithful character of R (of order coprime to p) and ρ a faithful character of H (of order a power of p). Let h be a generator of H and assume that $\text{ord}(h) = p^n$, for some $n \in \mathbb{Z}_{\geq 0}$. Let $e_\varphi := 1/|F| \sum_{f \in F} \varphi(f) \cdot f^{-1}$ be the idempotent associated to φ . Note that $e_\varphi \in \mathcal{O}[F]$. The following definition of “character μ -invariants” of \mathfrak{X}_S is due to Greenberg (see [21], pp. 656–657.)

Definition 5.8 (Greenberg). *Let $\psi \in \widehat{G}(\mathbb{C}_p)$ be an even character. Then $\mu_{\psi, \mathcal{O}}(\mathfrak{X}_S)$ is defined to be the μ -invariant $\mu_{\mathcal{O}}(\mathfrak{X}_S^\psi)$ of the $\mathcal{O}[[\Gamma]]$ -module*

$$\mathfrak{X}_S^\psi := \begin{cases} (h^{p^{n-1}} - 1)e_\varphi(\mathfrak{X}_{K_\psi} \otimes_{\mathbb{Z}_p} \mathcal{O}), & \text{if } n \geq 1; \\ e_\varphi(\mathfrak{X}_{K_\psi} \otimes_{\mathbb{Z}_p} \mathcal{O}), & \text{otherwise.} \end{cases}$$

Note that if the character ψ has order coprime to p , i.e. $\psi = \varphi$ and $n = 0$ in the above notation, then we have

$$e_\varphi(\mathfrak{X}_{K_\psi} \otimes_{\mathbb{Z}_p} \mathcal{O}) = \{x \in (\mathfrak{X}_{K_\psi} \otimes_{\mathbb{Z}_p} \mathcal{O}) \mid gx = \psi(g)x, \forall g \in G(K_\psi/k)\}.$$

This reconciles the definition above with Wiles's definition (see [34], p. 497) of “character μ -invariants” for characters of order coprime to p . (See also Definition 11.6.14 in [21].)

The following is the statement of the classical (non-equivariant) Main Conjecture for the module \mathfrak{X}_S^+ , proved by Wiles in [34]. Below, we denote by \mathfrak{m}_γ the multiplication-by- γ automorphism of all the relevant vector spaces.

Theorem 5.9 (Wiles). *For every even character $\psi \in \widehat{G}(\mathbb{C})$, we have*

$$G_{\psi, S}(t) \sim \pi^{\mu_{\psi, \mathcal{O}}(\mathfrak{X}_S)} \cdot \det_{Q(\mathcal{O})}((t+1) - \mathfrak{m}_\gamma \mid e_\psi(\mathfrak{X}_S \otimes_{\mathbb{Z}_p} Q(\mathcal{O}))),$$

where \sim denotes association in divisibility in the ring $\mathcal{O}[[t]]$ and e_ψ is the idempotent associated to ψ in $Q(\mathcal{O})[G]$.

Proof. (Sketch.) For characters ψ of order coprime to p , the statement above is the combination of Theorems 1.3 and 1.4 in [34]. For arbitrary characters, [34] contains a proof of the statement above only up to a power of π (Theorem 1.3. loc.cit.) However, once one has the right definition of $\mu_{\psi, \mathcal{O}}(\mathfrak{X}_S)$ (see above), one can deduce the statement above for arbitrary characters ψ without much difficulty from Theorems 1.3 and 1.4 in [34] restricted to the case of the trivial character and base fields K_ψ and K_{ψ^p} , respectively (see Theorem 11.6.16 in [21] for a proof.) \square

Remark 5.10. *As proved by Greenberg (see Proposition 1 in [10]), we have an equality of monic polynomials in $\mathcal{O}[t]$*

$$\det_{Q(\mathcal{O})}((t+1) - \mathfrak{m}_\gamma \mid e_\psi(\mathfrak{X}_S \otimes_{\mathbb{Z}_p} Q(\mathcal{O}))) = \det_{Q(\mathcal{O})}((t+1) - \mathfrak{m}_\gamma \mid e_\psi(\mathfrak{X}_{K_\psi} \otimes_{\mathbb{Z}_p} Q(\mathcal{O}))),$$

for all even ψ . Also, it is easy to see that

$$e_\psi(\mathfrak{X}_{K_\psi} \otimes_{\mathbb{Z}_p} Q(\mathcal{O})) = \mathfrak{X}_S^\psi \otimes_{\mathcal{O}} Q(\mathcal{O}),$$

for all even ψ . Consequently, the right hand-side of the equivalence \sim in Theorem 5.9 is precisely the characteristic polynomial $\text{char}_{\mathcal{O}[[\Gamma]]}(\mathfrak{X}_S^\psi)$ of the $\mathcal{O}[[\Gamma]]$ -module \mathfrak{X}_S^ψ . On the other hand, by definition, \mathfrak{X}_S^ψ is an $\mathcal{O}[[\Gamma]]$ -submodule of $(\mathfrak{X}_{K_\psi} \otimes_{\mathbb{Z}_p} \mathcal{O})$. As such, it is a torsion $\mathcal{O}[[\Gamma]]$ -module with no finite non trivial $\mathcal{O}[[\Gamma]]$ -submodules (a classical theorem of Iwasawa.) Consequently, $\text{pd}_{\mathcal{O}[[\Gamma]]} \mathfrak{X}_S^\psi = 1$ (see Lemma 4.7) and its first Fitting ideal $\text{Fit}_{\mathcal{O}[[\Gamma]]}(\mathfrak{X}_S^\psi)$ is principal, generated by $\text{char}_{\mathcal{O}[[\Gamma]]}(\mathfrak{X}_S^\psi)$ (see Lemma 2.4 in [23] and also Proposition 7.2(1) in the Appendix.) This leads to the following equivalent, and perhaps more elegant formulation of Theorem 5.9:

$$\text{Fit}_{\mathcal{O}[[\Gamma]]}(\mathfrak{X}_S^\psi) = (G_{\psi, S}(t)), \quad \text{for all even } \psi,$$

where $(G_{\psi, S}(t))$ denotes the principal ideal of $\mathcal{O}[[\Gamma]]$ generated by $G_{\psi, S}(t)$.

In light of this reformulation, it would be natural to expect that a G -equivariant refinement of Theorem 5.9 would give the Fitting ideal $\text{Fit}_{\mathbb{Z}_p[[\mathcal{G}]]^+}(\mathfrak{X}_S^+)$ in terms of an equivariant p -adic L -function. Unfortunately, in general, the $\mathbb{Z}_p[[\mathcal{G}]]^+$ -module \mathfrak{X}_S^+ has infinite projective dimension, which makes its Fitting ideal non-principal and difficult to compute. This is the main reason why the module \mathfrak{X}_S^+ is replaced with $T_p(\mathcal{M}_{S,\mathcal{T}}^{\mathcal{K}})^-$, a $\mathbb{Z}_p[[\mathcal{G}]]^-$ -module of projective dimension 1 (see Theorem 4.6 above), whose Fitting ideal is principal (see Proposition 7.2(1) in the Appendix). Moreover, $T_p(\mathcal{M}_{S,\mathcal{T}}^{\mathcal{K}})^-$ has a quotient isomorphic to $(\mathfrak{X}_S^+)^*(1)$ (see Remark 3.10.)

Corollary 5.11. *If the μ -invariant $\mu_{\mathbb{Z}_p}(\mathfrak{X}_S^+)$ of the $\mathbb{Z}_p[[\Gamma]]$ -module \mathfrak{X}_S^+ is 0, then the following hold for all even characters $\psi \in \widehat{G}(\mathbb{C}_p)$.*

- (1) $\mu_{\psi,\mathcal{O}}(\mathfrak{X}_S) = 0$.
- (2) $G_{\psi,S}(t) \sim \det_{Q(\mathcal{O})}((t+1) - \mathfrak{m}_\gamma \mid e_\psi(\mathfrak{X}_S \otimes_{\mathbb{Z}_p} Q(\mathcal{O})))$ in $\mathcal{O}[[t]]$.

Proof. (2) is a consequence of (1) and Theorem 5.9. Part (1) is proved as follows. It is easily seen that Galois restriction gives an isomorphism of $\mathbb{Z}_p[[\mathcal{G}]]$ -modules $\mathfrak{X}_S^+ \simeq \mathfrak{X}_{K^+}$, where K^+ is the maximal real subfield of K . Also, since ψ is even, we have $K_\psi \subseteq K^+$ and Galois restriction leads to an exact sequence of $\mathbb{Z}_p[[\mathcal{G}]]$ -modules

$$\mathfrak{X}_{K^+} \longrightarrow \mathfrak{X}_{K_\psi} \longrightarrow G(K_\psi \cap K'/k) \longrightarrow 1.$$

Since $G(K_\psi \cap K'/k)$ is finite and $\mu_{\mathbb{Z}_p}(\mathfrak{X}_S^+) = 0$, we have $\mu_{\mathbb{Z}_p}(\mathfrak{X}_{K_\psi}) = 0$. Consequently, we have $\mu_{\mathcal{O}}(\mathfrak{X}_{K_\psi} \otimes_{\mathbb{Z}_p} \mathcal{O}) = 0$. Therefore $\mu_{\mathcal{O}}(e_\psi(\mathfrak{X}_{K_\psi} \otimes_{\mathbb{Z}_p} \mathcal{O})) = 0$ and $\mu_{\psi,\mathcal{O}}(\mathfrak{X}_S) = 0$. We have used the fact that the μ -invariant of a quotient or submodule of an Iwasawa module is at most equal to that of the module itself. \square

5.2. The relevant equivariant p -adic L -functions. For simplicity, we will continue to assume that $\mu_p \subseteq K$, unless otherwise stated. If R is a commutative ring with 1, we denote by $Q(R)$ the total ring of fractions of R . By abusing notation, we denote by $\iota, t_m : Q(\mathcal{O}[[\mathcal{G}]]) \simeq Q(\mathcal{O}[[\mathcal{G}]])$ the unique $Q(\mathcal{O})$ -algebra automorphisms obtained by extending to $Q(\mathcal{O})[[\mathcal{G}]]$ the \mathbb{Z}_p -algebra automorphisms $\iota, t_m : \mathbb{Z}_p[[\mathcal{G}]] \simeq \mathbb{Z}_p[[\mathcal{G}]]$ defined in §7.2 of the Appendix, for all $m \in \mathbb{Z}$.

As usual, we freely identify $\mathcal{O}[[\mathcal{G}]]$ and $Q(\mathcal{O}[[\mathcal{G}]])$ with $\mathcal{O}[G][[t]]$ and $Q(\mathcal{O}[G][[t]])$, respectively, via the first isomorphism in (21). By abusing notation once again, we let $\psi : \mathcal{O}[G][[t]] \rightarrow \mathcal{O}[[t]]$ denote the unique $\mathcal{O}[[t]]$ -algebra morphism extending $\psi : G \rightarrow \mathcal{O}$, for all $\psi \in \widehat{G}(\mathbb{C}_p)$.

Remark 5.12. *Note that the non zero-divisors of $\mathcal{O}[G][[t]]$ (respectively $\mathcal{O}[G]$) are precisely those elements $f \in \mathcal{O}[G][[t]]$ (respectively $f \in \mathcal{O}[G]$) with the property that $\psi(f) \neq 0$ in $\mathcal{O}[[t]]$ (respectively in \mathcal{O}), for all $\psi \in \widehat{G}(\mathbb{C}_p)$.*

We consider the following power series in $\frac{1}{|G|}\mathcal{O}[G][[t]]$:

$$G_S(t) := \sum_{\psi \in \widehat{G}(\mathbb{C}_p)} G_{\psi,S}(t) \cdot e_\psi, \quad H_S(t) := \sum_{\psi \in \widehat{G}(\mathbb{C}_p)} H_{\psi,S}(t) \cdot e_\psi = t \cdot e_{1_G} + (1 - e_{1_G}).$$

Observe that, for all $m \in \mathbb{Z}$, we have equalities

$$\iota(e_\psi) = e_{\psi^{-1}}, \quad t_m(e_\psi) = e_{\psi\omega^{-m}}, \quad \iota(t) = (1+t)^{-1} - 1, \quad t_m(t) = u^m(1+t) - 1,$$

for all characters $\psi \in \widehat{G}(\mathbb{C}_p)$. Consequently, we have the following, for all $m \in \mathbb{Z}$.

$$\begin{aligned} (\iota \circ t_m)(G_S(t)) &= \sum_{\psi \in \widehat{G}(\mathbb{C}_p)} G_{\psi^{-1}\omega^m, S}(u^m(1+t)^{-1} - 1) \cdot e_\psi, \\ (\iota \circ t_m)(H_S(t)) &= (u^m(1+t)^{-1} - 1) \cdot e_{\omega^m} + (1 - e_{\omega^m}). \end{aligned}$$

For every $n \in \mathbb{Z}_{\geq 0}$, we let K_n denote the unique field, with $K \subseteq K_n \subseteq \mathcal{K}$ and $[K_n : K] = p^n$. We let $G_n := G(K_n/k)$ and note that $G_n \simeq G \times \Gamma/\Gamma^{p^n}$. We denote by $\pi_n : \mathcal{O}[[\mathcal{G}]] \rightarrow \mathcal{O}[G_n]$ the usual (surjective) \mathcal{O} -algebra morphism induced by Galois restriction. By applying the remark above to the group rings $\mathcal{O}[[\mathcal{G}]]$ and $\mathcal{O}[G_n]$, it is easily seen that $|G|(\iota \circ t_m)(H_S)$ is not a zero-divisor in $\mathcal{O}[[\mathcal{G}]]$, for all $m \neq 0$. Moreover, $\pi_n(|G|(\iota \circ t_m)(H_S))$ is not a zero-divisor in $\mathcal{O}[G_n]$, for all n . Let

$$\Upsilon := \{f \in \mathcal{O}[[\mathcal{G}]] \mid \pi_n(f) \text{ not a zero-divisor in } \mathcal{O}[G_n], \forall n\}, \quad \Upsilon_n := \pi_n(\Upsilon).$$

It is easily proved that Υ_n is in fact the set of all non zero-divisors of $\mathcal{O}[G_n]$. As a consequence, we have an equality $\Upsilon_n^{-1}\mathcal{O}[G_n] = Q(\mathcal{O}[G_n]) = Q(\mathcal{O})[G_n]$. The morphisms π_n above can be uniquely extended to surjective $Q(\mathcal{O})$ -algebra morphisms (for simplicity, also denoted) $\pi_n : \Upsilon^{-1}\mathcal{O}[[\mathcal{G}]] \rightarrow Q(\mathcal{O}[G_n])$, whose projective limit gives an injective $Q(\mathcal{O})$ -algebra morphism

$$\Upsilon^{-1}\mathcal{O}[[\mathcal{G}]] \hookrightarrow \varprojlim_n Q(\mathcal{O}[G_n]).$$

Next, we consider the element

$$g_S := \frac{(\iota \circ t_1)(G_S(t))}{(\iota \circ t_1)(H_S(t))} = \frac{|G|(\iota \circ t_1)(G_S(t))}{|G|(\iota \circ t_1)(H_S(t))} \in \Upsilon^{-1}\mathcal{O}[[\mathcal{G}]],$$

and describe its images $\pi_n(g_S)$ in $Q(\mathcal{O}[G_n])$, for all n . In order to do that, let

$$\Theta_S^{(n)}(s) := \Theta_{S, K_n/k}(s), \quad \forall n \in \mathbb{Z}_{\geq 0}$$

be the S -incomplete G_n -equivariant L -function associated to K_n/k . According to Siegel's Theorem 5.1, we have

$$\Theta_S^{(n)}(1-m) \in \mathbb{Q}[G_n] \subseteq Q(\mathcal{O}[G_n]), \quad \forall n \in \mathbb{Z}_{\geq 0}, \quad \forall m \in \mathbb{Z}_{\geq 1}.$$

Moreover, the inflation property of Artin L -functions implies that

$$(\Theta_S^{(n)}(1-m))_n \in \varprojlim_n Q(\mathcal{O}[G_n]), \quad \forall m \in \mathbb{Z}_{\geq 1}.$$

We let $\Theta_S^{(\infty)}(1-m) := (\Theta_S^{(n)}(1-m))_n$ and view it as an element of $\varprojlim_n Q(\mathcal{O}[G_n])$.

Lemma 5.13. *For all $m \in \mathbb{Z}_{\geq 1}$, we have.*

- (1) $t_{1-m}(g_S) \in \Upsilon^{-1}\mathcal{O}[[\mathcal{G}]]$.
- (2) $\Theta_S^{(\infty)}(1-m) = t_{1-m}(g_S)$ in $\Upsilon^{-1}\mathcal{O}[[\mathcal{G}]]$.

Proof. Part (1) follows from the observation that the element

$$t_{1-m}(|G|(\iota \circ t_1)(H_S)) = |G|(\iota \circ t_m)(H_S)$$

belongs to Υ , for all $m \in \mathbb{Z}_{\geq 1}$. Part (2) is equivalent to

$$\pi_n(t_{1-m}(g_S)) = \Theta_S^{(n)}(1-m),$$

for all $m \in \mathbb{Z}_{\geq 1}$ and all $n \in \mathbb{Z}_{\geq 0}$. This is Proposition 4.1 in [23]. Note that in loc.cit. our power series $(\iota \circ t_1)(G_S(t))$ and $(\iota \circ t_1)(H_S(t))$ are denoted $G_S(t)$ and $H_S(t)$, respectively. However, g_S denotes the same element in $\mathcal{Y}^{-1}\mathcal{O}[[\mathcal{G}]]$. \square

Next, we use the additional set of primes T to eliminate the denominators of the elements $t_{1-m}(g_S)$ in $\mathcal{Y}^{-1}\mathcal{O}[[\mathcal{G}]]$, for all $m \in \mathbb{Z}_{\geq 1}$. For that purpose, we consider

$$\delta_T^{(n)} := \delta_{T, K^{(n)}/k}, \quad \Theta_{S,T}^{(n)} := \Theta_{S, T, K^{(n)}/k} = \delta_{T, K^{(n)}/k} \cdot \Theta_{S, K^{(n)}/k},$$

viewed as holomorphic functions $\mathbb{C} \rightarrow \mathbb{C}[G_n]$, for all $n \in \mathbb{Z}_{\geq 0}$. Note that we have

$$\delta_T^{(\infty)}(1-m) := (\delta_T^{(n)}(1-m))_n \in \varprojlim_n \mathbb{Z}_p[G_n] = \mathbb{Z}_p[[\mathcal{G}]],$$

for all $m \in \mathbb{Z}_{\geq 1}$, as an immediate consequence of the definition of the $\delta_T^{(n)}$'s. Consequently, since $T \cap S_p = \emptyset$, Remark 5.5 implies that

$$(\Theta_{S,T}^{(n)}(1-m))_n \in \varprojlim_n \mathbb{Z}_p[G_n] = \mathbb{Z}_p[[\mathcal{G}]], \quad \forall m \in \mathbb{Z}_{\geq 1}.$$

We let $\Theta_{S,T}^{(\infty)}(1-m) := (\Theta_{S,T}^{(n)}(1-m))_n$ and view these as elements in $\mathbb{Z}_p[[\mathcal{G}]]$.

Lemma 5.14. *For all $m \in \mathbb{Z}_{\geq 1}$, we have the following equalities in $\mathbb{Z}_p[[\mathcal{G}]]$.*

- (1) $\delta_T^{(\infty)}(1-m) = t_{1-m}(\delta_T^{(\infty)}(0))$.
- (2) $\Theta_{S,T}^{(\infty)}(1-m) = t_{1-m}(\delta_T^{(\infty)}(0)) \cdot t_{1-m}(g_S) = t_{1-m}(\Theta_{S,T}^{(\infty)}(0))$.

Proof. Let m be as above. From the definitions, we have

$$\delta_T^{(\infty)}(1-m) = \prod_{v \in T} (1 - (\sigma_v^{(\infty)})^{-1} \cdot \mathbf{N}v^m),$$

where $\sigma_v^{(\infty)}$ is the Frobenius morphism associated to v in \mathcal{G} . Now, since $T \cap S_p = \emptyset$, we have $c(\sigma_v^{(\infty)}) = \mathbf{N}v$, for all $v \in T$. Combined with the last displayed equality, this implies part (1) of the Lemma. Part (2) is a direct consequence of part (1) combined with Lemma 5.13(2). \square

Corollary 5.15. *We have the following.*

- (1) $g_S \in \mathcal{Y}_{\mathbb{Z}_p}^{-1}\mathbb{Z}_p[[\mathcal{G}]]$, where $\mathcal{Y}_{\mathbb{Z}_p} := \mathcal{Y} \cap \mathbb{Z}_p[[\mathcal{G}]]$.
- (2) $G_S \in 1/|G|\mathbb{Z}_p[[\mathcal{G}]]$.

Proof. Part (1) follows from Lemma 5.14(2) with $m = 1$ and the obvious observation that $\delta_T^{(\infty)}(0) \in \mathcal{Y}_{\mathbb{Z}_p}$. Part (2) is an immediate consequence of part (1) and the fact that $H_S \in 1/|G|\mathbb{Z}_p[[\mathcal{G}]]$. \square

Definition 5.16. *The S -incomplete, respectively S -incomplete T -modified, G -equivariant p -adic L -functions $\Theta_S^{(\infty)} \in \mathcal{Y}_{\mathbb{Z}_p}^{-1}\mathbb{Z}_p[[\mathcal{G}]]$ and $\Theta_{S,T}^{(\infty)} \in \mathbb{Z}_p[[\mathcal{G}]]$ associated to $(K/k, S, T, p)$ are defined by*

$$\Theta_S^{(\infty)} := \Theta_S^{(\infty)}(0) = g_S, \quad \Theta_{S,T}^{(\infty)} := \Theta_{S,T}^{(\infty)}(0).$$

For consistency, we let $\delta_T^{(\infty)} := (\delta_T^{(n)}(0))_n \in \mathbb{Z}_p[[\mathcal{G}]]$.

Remark 5.17. Note that, by Lemmas 5.13 and 5.14, we have

$$\Theta_{S,T}^{(\infty)} = \delta_T^{(\infty)} \cdot \Theta_S^{(\infty)}$$

for all $m \in \mathbb{Z}_{\geq 1}$. Obviously, the link between $\Theta_{S,T}^{(\infty)}$ and the classical p -adic L -functions $L_p(\psi, s)$ defined in (23) is given by

$$(24) \quad \chi(\Theta_{S,T}^{(\infty)}) = \chi(\delta_T^{(\infty)}) \cdot \frac{G_{\chi^{-1}\omega, S}(u(1+t)^{-1} - 1)}{H_{\chi^{-1}\omega, S}(u(1+t)^{-1} - 1)},$$

for all characters $\chi \in \widehat{G}(\mathbb{C}_p)$. Also, note that since for all odd characters $\psi \in \widehat{G}_n(\mathbb{C}_p)$ we have $G_{\psi, S} = 0$ (see Remark 5.7) and p is odd, we have

$$\Theta_S^{(\infty)} \in \frac{1}{2}(1-j) \cdot \Upsilon_{\mathbb{Z}_p}^{-1} \mathbb{Z}_p[[\mathcal{G}]], \quad \Theta_{S,T}^{(\infty)} \in \frac{1}{2}(1-j) \cdot \mathbb{Z}_p[[\mathcal{G}]].$$

In what follows, we view $\Theta_{S,T}^{(\infty)}$ and $\Theta_S^{(\infty)}$ as elements of both $1/2(1-j)\mathbb{Z}_p[[\mathcal{G}]]$ and $\mathbb{Z}_p[[\mathcal{G}]]^- = \mathbb{Z}_p[[\mathcal{G}]]/(1+j)$ via the ring isomorphism $1/2(1-j)\mathbb{Z}_p[[\mathcal{G}]] \simeq \mathbb{Z}_p[[\mathcal{G}]]^-$ given by reduction modulo $(1+j)$.

Remark 5.18. In this section we assumed that $\mu_p \subseteq K$ (i.e. $\mu_{p^\infty} \subseteq K$). This hypothesis was used whenever the twisting automorphisms t_m of $\mathcal{O}[[\mathcal{G}]]$ with respect to the various powers c^m of the p -cyclotomic character c of \mathcal{G} were needed.

However, even if $\mu_p \not\subseteq K$, we can still construct any object $*$ in the list $G_S(t)$, $H_S(t)$, $\Theta_S^{(n)}(s)$, $\delta_T^{(n)}(s)$, $\Theta_{S,T}^{(n)}(s)$ for the data $(K/k, S, T, p)$, just as above. Let $\tilde{K} := K(\mu_p)$, $\tilde{K} = K(\mu_p)$, $\tilde{G} := G(\tilde{K}/k)$, $\tilde{G}_n := G(\tilde{K}_n/k)$, $\tilde{\mathcal{G}} := G(\tilde{K}/k)$. Further, let $\tilde{*}$ denote the analogue of $*$ for $(\tilde{K}/k, S, T, p)$. By abuse of notation, let $\pi : \mathbb{C}[\tilde{G}_n] \twoheadrightarrow \mathbb{C}[G_n]$, $\pi : \mathcal{O}[[\tilde{\mathcal{G}}]] \twoheadrightarrow \mathcal{O}[[\mathcal{G}]]$, etc., denote the \mathcal{O} -algebra morphisms induced by Galois restriction. The inflation property of Artin L -functions immediately implies that

$$\pi(\tilde{*}) = *,$$

for all $*$ as above. Consequently, we can construct equivariant p -adic L -functions $\Theta_S^{(\infty)} := (\Theta_S^{(n)}(0))_n \in \Upsilon_{\mathbb{Z}_p}^{-1} \mathbb{Z}_p[[\mathcal{G}]]^-$ and $\Theta_{S,T}^{(\infty)} := (\Theta_{S,T}^{(n)}(0))_n \in \mathbb{Z}_p[[\mathcal{G}]]^-$, for $(K/k, S, T, p)$. Obviously, these will satisfy

$$\pi(\tilde{\Theta}_{S,T}^{(\infty)}) = \Theta_{S,T}^{(\infty)}.$$

5.3. The Proof of the Equivariant Main Conjecture. In this section, we prove Theorem 5.6. Consequently, we work under the hypothesis $\mu_K = 0$. In particular, we have $\mu_{\mathbb{Z}_p}(\mathfrak{X}_S^+) = 0$. (Recall from classical Iwasawa theory that $\mu_{\mathbb{Z}_p}(\mathfrak{X}_S^+) = \mu_{K,p}^-$, for all sets S as above.)

Lemma 5.19. It suffices to prove Theorem 5.6 under the assumption that $\mu_p \subseteq K$.

Proof. Assume that $\mu_p \not\subseteq K$. With notations as in remark above, let $\Delta := G(\tilde{K}/K)$. Also let $\mathcal{M} := \mathcal{M}_{\tilde{S}, \tilde{\mathcal{T}}}^K$ and $\tilde{\mathcal{M}} := \mathcal{M}_{\tilde{S}, \tilde{\mathcal{T}}}^{\tilde{K}}$, where \tilde{S} and $\tilde{\mathcal{T}}$ are the sets of primes in \tilde{K} sitting above primes in \mathcal{S} and \mathcal{T} , respectively. Since

$$\ker(\pi : \mathbb{Z}_p[[\tilde{\mathcal{G}}]]^- \twoheadrightarrow \mathbb{Z}_p[[\mathcal{G}]]^-) = I_\Delta \mathbb{Z}_p[[\tilde{\mathcal{G}}]]^-,$$

where I_Δ is the augmentation ideal in $\mathbb{Z}_p[\Delta]$, Corollary 4.8 implies that we have isomorphisms of $\mathbb{Z}_p[[\mathcal{G}]]^-$ -modules

$$T_p(\mathcal{M})^- \simeq T_p(\widetilde{\mathcal{M}})^- / I_\Delta T_p(\widetilde{\mathcal{M}})^- \simeq T_p(\widetilde{\mathcal{M}})^- \otimes_{\mathbb{Z}_p[[\widetilde{\mathcal{G}}]]^-} \mathbb{Z}_p[[\mathcal{G}]]^-.$$

Consequently, the second equality in (44) (Appendix) implies that we have

$$\mathrm{Fit}_{\mathbb{Z}_p[[\mathcal{G}]]^-} T_p(\mathcal{M})^- = \pi(\mathrm{Fit}_{\mathbb{Z}_p[[\widetilde{\mathcal{G}}]]^-} T_p(\widetilde{\mathcal{M}})^-).$$

Now the last equality in Remark 5.3 shows that Theorem 5.6 for the data $(\widetilde{\mathcal{K}}/k, S, T, p)$ implies the same result for the data $(\mathcal{K}/k, S, T, p)$. \square

As a consequence of the above Lemma, we may and will assume that $\mu_p \subseteq \mathcal{K}$ throughout the rest of this section. Now, note that Theorem 4.6(1) shows that the $\mathbb{Z}_p[G]^-$ -module $T_p(\mathcal{M}_{S,\mathcal{T}}^\mathcal{K})^-$ is projective. Consequently, if we apply Proposition 7.2(1) (see Appendix) for the semi-local ring $R := \mathbb{Z}_p[G]^-$, $g := \gamma$ and the $R[[\Gamma]]^-$ -module $M := T_p(\mathcal{M}_{S,\mathcal{T}}^\mathcal{K})^-$, we conclude that Theorem 5.6 is equivalent to the following association in divisibility in $\mathbb{Z}_p[[\mathcal{G}]]^- \simeq \mathbb{Z}_p[G]^-[[t]]$:

$$(25) \quad \Theta_{S,T}^{(\infty)} \sim \det_{\mathbb{Z}_p[G]^-}((t+1) - \mathbf{m}_\gamma \mid T_p(\mathcal{M}_{S,\mathcal{T}}^\mathcal{K})^-).$$

Lemma 5.20. *For all odd characters $\chi \in \widehat{G}(\mathbb{C}_p)$, we have the following in $\mathcal{O}[[t]]$.*

- (1) $G_{\chi^{-1}\omega,S}(u(1+t)^{-1}-1) \sim \chi \left(\det_{Q(\mathcal{O})[G]}((t+1) - \mathbf{m}_\gamma \mid T_p(\mathcal{M}_{S,\emptyset}^\mathcal{K})^- \otimes_{\mathbb{Z}_p} Q(\mathcal{O})) \right).$
- (2) $\mu(G_{\chi^{-1}\omega,S}(u(1+t)^{-1}-1)) = 0$, with notations as in Corollary 7.9.

Proof. (1) Combine Corollary 5.11(2) and Lemma 7.5(3), applied to the lattice $\mathcal{L} := \mathfrak{X}_S^+ \otimes_{\mathbb{Z}_p} \mathcal{O}$ and $n = 1$ to conclude that

$$G_{\chi^{-1}\omega,S}(u(1+t)^{-1}-1) \sim \chi \left(\det_{Q(\mathcal{O})[G]}((t+1) - \mathbf{m}_\gamma \mid (\mathfrak{X}_S^+)^*(1) \otimes_{\mathbb{Z}_p} Q(\mathcal{O})) \right).$$

Now use the $\mathbb{Z}_p[[\mathcal{G}]]^-$ -module isomorphism $T_p(\mathcal{M}_{S,\emptyset}^\mathcal{K})^- \simeq (\mathfrak{X}_S^+)^*(1)$ given by Lemma 3.9 to conclude the proof of (1).

(2) This follows from (1) and the fact that the right-hand-side in (1) is a monic polynomial in $\mathcal{O}[t]$ (see Lemma 7.5 in the Appendix.) \square

Lemma 5.21. *If $Q(\mathcal{O})$ is endowed with the trivial \mathcal{G} -action, then:*

- (1) $H_S(t) = \det_{Q(\mathcal{O})[G]}((t+1) - \mathbf{m}_\gamma \mid Q(\mathcal{O})).$
- (2) *For all odd $\chi \in \widehat{G}(\mathbb{C}_p)$, we have an association in divisibility in $\mathcal{O}[[t]]$:*

$$H_{\chi^{-1}\omega,S}(u(1+t)^{-1}-1) \sim \chi(\det_{Q(\mathcal{O})[G]}((t+1) - \mathbf{m}_\gamma \mid Q(\mathcal{O})(1))).$$
- (3) *For all odd $\chi \in \widehat{G}(\mathbb{C}_p)$, we have $\mu(H_{\chi^{-1}\omega,S}(u(1+t)^{-1}-1)) = 0$, with notations as in Corollary 7.9.*

Proof. Part (1) is clear from the equality $H_S(t) := t \cdot e_{1_G} + (1 - e_{1_G})$. Part (2) is a direct consequence of part (1), and Lemma 7.5(3) applied to $\mathcal{L} := \mathcal{O}$ (with trivial \mathcal{G} -action), $V = Q(\mathcal{O})$, and $n = 1$. Observe that $Q(\mathcal{O})^*(1) \simeq Q(\mathcal{O})(1)$. Part (3) follows from (2) and the fact that the right-hand-side in (2) is a monic polynomial in $\mathcal{O}[t]$ (see Lemma 7.5 in the Appendix.) \square

Lemma 5.22. *The $\mathbb{Z}_p[[\mathcal{G}]]^-$ -module $T_p(\Delta_{\mathcal{K},\mathcal{T}})$ satisfies the following.*

- (1) $T_p(\Delta_{\mathcal{K}, \mathcal{T}}) \simeq \bigoplus_{v \in T} \mathbb{Z}_p[[\mathcal{G}]]/(\delta_v^{(\infty)})$, where $\delta_v^{(\infty)} := (1 - (\sigma_v^{(\infty)})^{-1} \cdot \mathbf{N}v)$.
- (2) $\text{Fit}_{\mathbb{Z}_p[[\mathcal{G}]]} T_p(\Delta_{\mathcal{K}, \mathcal{T}}) = (\delta_T^{(\infty)})$.
- (3) $\text{pd}_{\mathbb{Z}_p[[\mathcal{G}]]} T_p(\Delta_{\mathcal{K}, \mathcal{T}}) = 1$ and $\text{pd}_{\mathbb{Z}_p[G]} T_p(\Delta_{\mathcal{K}, \mathcal{T}}) = 0$.
- (4) $\delta_T^{(\infty)} \sim \det_{\mathbb{Z}_p[G]}((t+1) - \mathbf{m}_\gamma \mid T_p(\Delta_{\mathcal{K}, \mathcal{T}}))$ in $\mathbb{Z}_p[[\mathcal{G}]]$
- (5) $\mu(\chi(\delta_T^{(\infty)})) = 0$, for all $\chi \in \widehat{G}(\mathbb{C}_p)$.

Proof. (1) For every $v \in T$, let us fix a prime $w(v) \in \mathcal{T}$ sitting above v . Then, we have an obvious isomorphism of $\mathbb{Z}[[\mathcal{G}]]$ -modules

$$\Delta_{\mathcal{K}, \mathcal{T}} \simeq \bigoplus_{v \in T} (\kappa(w(v))^\times \otimes_{\mathbb{Z}[[\mathcal{G}_v]]} \mathbb{Z}[[\mathcal{G}]]),$$

where \mathcal{G}_v is the decomposition group associated to v in \mathcal{G} (topologically generated by $\sigma_v^{(\infty)}$) and $\kappa(w(v))$ denotes, as usual, the residue field associated to $w(v)$. This induces an isomorphism of $\mathbb{Z}_p[[\mathcal{G}]]$ -modules

$$T_p(\Delta_{\mathcal{K}, \mathcal{T}}) \simeq \bigoplus_{v \in T} (T_p(\kappa(w(v))^\times) \otimes_{\mathbb{Z}_p[[\mathcal{G}_v]]} \mathbb{Z}_p[[\mathcal{G}]]).$$

The isomorphism above combined with Lemma 7.4(4) (see Appendix) applied to the extension $\kappa(w(v))/\kappa(v)$ of Galois group \mathcal{G}_v , for $q := \text{card}(\kappa(v))$ and $\sigma_q = \sigma_v^{(\infty)}$ concludes the proof of (1).

(2) is a direct consequence of (1) and the definition of the Fitting ideal.

(3) Note that if $\alpha \in \mathbb{Z}_p$, such that $\sigma_v^{(\infty)} = \gamma^\alpha$ and σ_v is the Frobenius associated to v in G , then we have

$$\delta_v^{(\infty)} \sim (\sigma_v \cdot (t+1)^\alpha - \mathbf{N}v) \quad \text{in } \mathbb{Z}_p[[\mathcal{G}]] \simeq \mathbb{Z}_p[G][[t]].$$

Consequently, $\chi(\delta_v^{(\infty)}) = (\chi(\sigma_v)(1+t)^\alpha - \mathbf{N}v)$ is not a zero divisor in $\mathbb{Z}_p(\chi)[[t]]$, for all $\chi \in \widehat{G}(\mathbb{C}_p)$ and therefore $\delta_v^{(\infty)}$ is not a zero divisor in $\mathbb{Z}_p[[\mathcal{G}]]$, for all $v \in T$. This observation combined with the isomorphism in part (1) leads to the equality $\text{pd}_{\mathbb{Z}_p[[\mathcal{G}]]} T_p(\Delta_{\mathcal{K}, \mathcal{T}}) = 1$. Now, in order to obtain the equality $\text{pd}_{\mathbb{Z}_p[G]} T_p(\Delta_{\mathcal{K}, \mathcal{T}}) = 0$, one applies Lemma 4.7 for $M := T_p(\Delta_{\mathcal{K}, \mathcal{T}})$ and $H := G$, keeping in mind that $T_p(\Delta_{\mathcal{K}, \mathcal{T}})$ is \mathbb{Z}_p -free.

(4) This is a direct consequence of (2) and (3) combined with Proposition 7.2(1) applied for $R := \mathbb{Z}_p[G]$, $g := \gamma$ and the $R[[\Gamma]] = \mathbb{Z}_p[[\mathcal{G}]]$ -module $M := T_p(\Delta_{\mathcal{K}, \mathcal{T}})$.

(5) This is a direct consequence of (4) and the fact that the right-hand-side in (4) is a monic polynomial in $\mathbb{Z}_p[G][[t]]$ (see Remark 7.1 in the Appendix.) \square

Remark 5.23. *Note that the proofs of (1), (2) and the first equality in (3) of the above Lemma are independent on our assumption that $\mathcal{G} \simeq G \times \Gamma$.*

Now, we are ready to prove (25) above. Let

$$\Theta := \Theta_{S, T}^{(\infty)}, \quad F := \det_{\mathbb{Z}_p[G]^-}((t+1) - \mathbf{m}_\gamma \mid T_p(\mathcal{M}_{S, \mathcal{T}}^\kappa)^-).$$

We view Θ and F as power series in $\mathbb{Z}_p[G]^-[[t]]$ and plan on using Corollary 7.9 (see Appendix) to show that $\Theta \sim F$. Note that, in fact, F is a monic polynomial

in $\mathbb{Z}_p[G]^-[[t]]$ (see Remark 7.1 in the Appendix). Consequently, with notations as in Corollary 7.9, we have

$$(26) \quad \mu(\chi(F)) = 0, \quad \forall \chi \in \widehat{G}(\mathbb{C}_p), \chi \text{ odd.}$$

Now, Lemma 5.20(2), Lemma 5.21(3) and Lemma 5.22(5) combined with equality (24) above show that

$$(27) \quad \mu(\chi(\Theta)) = 0, \quad \forall \chi \in \widehat{G}(\mathbb{C}_p), \chi \text{ odd.}$$

Now, exact sequence (15) combined with equality (24) and Lemma 5.20(1), Lemma 5.21(2) and Lemma 5.22(4) show that the following holds in $\mathcal{O}[[t]]$.

$$(28) \quad \chi(\Theta) \sim \chi(\det_{Q(\mathcal{O})[G]}((t+1) - \mathfrak{m}_\gamma \mid T_p(\mathcal{M}_{S,\mathcal{T}}^\mathcal{K})^- \otimes_{\mathbb{Z}_p} Q(\mathcal{O})) = \chi(F),$$

for all odd $\chi \in \widehat{G}(\mathbb{C}_p)$. Now, as a consequence of (26), (27) and (28), Corollary 7.9 in the Appendix leads to

$$\Theta \sim F \text{ in } \mathbb{Z}_p[G]^-[[t]],$$

which concludes the proof of (25) and that of Theorem 5.6. \square

Corollary 5.24. *Under the hypotheses of Theorem 5.6, we have*

$$\text{Fit}_{\mathbb{Z}_p[[\mathcal{G}]]}((T_p(\mathcal{M}_{S,\mathcal{T}}^\mathcal{K})^-)^*) = \text{Fit}_{\mathbb{Z}_p[[\mathcal{G}]]}(T_p(\mathcal{M}_{S,\mathcal{T}}^\mathcal{K})^-) = \left(\Theta_{S,T}^{(\infty)}, \frac{1}{2}(1+j) \right)$$

where $(T_p(\mathcal{M}_{S,\mathcal{T}}^\mathcal{K})^-)^* := \text{Hom}_{\mathbb{Z}_p}(T_p(\mathcal{M}_{S,\mathcal{T}}^\mathcal{K})^-, \mathbb{Z}_p)$ with the coinvariant \mathcal{G} -action.

Proof. Theorem 4.6(1) permits us to apply Proposition 7.2(3) (see Appendix) to $M := T_p(\mathcal{M}_{S,\mathcal{T}}^\mathcal{K})^-$. The latter implies the first equality above. The second equality is a direct consequence of Theorem 5.6. Note that in the statement of the Corollary $\Theta_{S,T}^{(\infty)}$ is viewed as an element of $\mathbb{Z}_p[[\mathcal{G}]]$. \square

6. APPLICATIONS OF THE EQUIVARIANT MAIN CONJECTURE

In this section, we give two application of Theorem 5.6 proved above: the first is a proof of a refinement of an imprimitive version of the Brumer–Stark Conjecture, away from its 2–primary part (see Theorem 6.5 below); the second is a proof of a refinement of the Coates–Sinnott Conjecture, away from its 2–primary part (see Theorem 6.11 and Corollary 6.19 below.) Of course, these theorems are proved under the assumption that the relevant classical Iwasawa μ –invariants vanish.

In what follows, if M is a $\mathbb{Z}_p[[\mathcal{H}]]$ –module, for some prime p and some abelian profinite group \mathcal{H} , then $M^\vee := \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ and $M^* := \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ denote its Pontrjagin and \mathbb{Z}_p –module dual, respectively. depending on the context, M^\vee and M^* will be endowed either with the covariant \mathcal{H} –action $(\sigma f)(m) := f(\sigma m)$ or the contravariant one $(\sigma f)(m) := f(\sigma^{-1}m)$, for all $m \in M$, $\sigma \in \mathcal{H}$, and all $f \in M^\vee$ and $f \in M^*$, respectively.

Throughout this section, we let K/k be an abelian extension of number fields of Galois group G and let S be a fixed finite set of primes in k , containing the set $S_{\text{ram}}(K/k) \cup S_\infty$. We will use the notations introduced at the beginning of §5.

6.1. The Brumer-Stark Conjecture. We fix a non-empty, finite set T of primes in k , such that $S \cap T = \emptyset$. We assume that T contains either at least two primes of distinct residual characteristics or a prime of residual characteristic coprime to $w_K := |\mu_K|$, where μ_K denotes the group of roots of unity in K . (It is easily seen that this last condition is equivalent to the non-existence of non trivial roots of unity in K which are congruent to 1 modulo all primes in T .)

For simplicity, we let $\Theta_{S,T} := \Theta_{S,T,K/k}$ denote the S -incomplete, T -modified G -equivariant L -function associated to $(K/k, S, T)$ at the beginning of §5. Note that, under our assumptions, Remark 5.5 for $m = 1$ gives

$$\Theta_{S,T}(0) \in \mathbb{Z}[G].$$

In order to simplify notations, we let S and T also denote the sets of primes in K sitting above primes in the original sets S and T , respectively. Similarly, depending on the context, S_∞ will denote the set of infinite primes in either K or k . As in §4.3 of [24], to the set of data (K, T) , one can associate a T -modified Arakelov class-group (Chow group) $\text{CH}^1(K)_T^0$, endowed with a natural $\mathbb{Z}[G]$ -module structure, as follows. First, one defines the divisor group

$$\text{Div}(K)_T := \left(\bigoplus_{w \notin S_\infty \cup T} \mathbb{Z} \cdot w \right) \bigoplus \left(\bigoplus_{w \in S_\infty} \mathbb{R} \cdot w \right),$$

where the direct sums are taken over all the primes w in K . Then, one defines

$$\deg_{K,T} : \text{Div}(K)_T \rightarrow \mathbb{R}$$

to be the unique degree map which is \mathbb{Z} -linear on the left direct summand and \mathbb{R} -linear on the right and satisfies

$$\deg_{K,T}(w) := \begin{cases} \log |\mathbf{N}w|, & \text{if } w \text{ is a finite prime;} \\ 1 & \text{if } w \text{ is an infinite prime.} \end{cases}$$

We let $\text{Div}(K)_T^0 := \ker(\deg_{K,T})$. Next, one defines a divisor map

$$\text{div}_K : K_T^\times \rightarrow \text{Div}(K)_T^0, \quad \text{div}_K(x) := \sum_{w \notin S_\infty \cup T} \text{ord}_w(x) \cdot w + \sum_{w \in S_\infty} (-\log |x|_w) \cdot w,$$

where ord_w and $|\cdot|_w$ denote the canonically normalized valuation and metric associated to w , respectively and K_T^\times denotes the subgroup of K^\times consisting of elements congruent to 1 modulo all primes in T . Finally, one defines

$$\text{CH}^1(K)_T^0 := \frac{\text{Div}(K)_T^0}{\text{div}_K(K_T^\times)}.$$

If endowed with the natural quotient topology, this is a compact group whose volume is the absolute value of the leading term $\zeta_{S,T,k}^*(0)$ at $s = 0$ of the modified zeta function $\zeta_{S,T,k} := \Theta_{S,T,k/k} : \mathbb{C} \rightarrow \mathbb{C}$ associated to k (see loc.cit.)

Remark 6.1. Note the difference, both in notation and definition, between the finite divisor group $\text{Div}_{K,T}$ and divisor map div_K defined in §2.2 and, respectively, the Arakelov divisor group $\text{Div}(K)_T$ and divisor map div_K defined above. The link between the finite T -modified class-group $C_{K,T}$ defined in §2.2 and the compact T -modified Arakelov class-group $\text{CH}^1(K)_T^0$ defined above is captured by the following

obvious exact sequence of $\mathbb{Z}[G]$ -modules.

$$0 \longrightarrow \frac{\mathrm{Div}(K)^0(S_\infty)}{\mathrm{div}_K(U_{K,T})} \longrightarrow \mathrm{CH}^1(K)_T^0 \longrightarrow C_{K,T} \longrightarrow 0.$$

Above, $\mathrm{Div}(K)^0(S_\infty) := \bigoplus_{v \in S_\infty} \mathbb{R} \cdot v$ denotes the \mathbb{R} -vector space of Arakelov divisors of degree 0 supported on S_∞ (denoted by $\mathbb{R}X_{S_\infty}$ in [24]) and $U_{K,T}$ denotes the group of units in K which are congruent to 1 modulo all the primes in T . (See §4.3 of [24] for the exact sequence above.)

In §4.3 of [24], we proved that the Brumer–Stark Conjecture $\mathbf{BrSt}(K/k, S)$ for the data $(K/k, S)$ (as stated, for example, in Chpt. IV, §6 of [32]) is equivalent to the following statement.

Conjecture 6.2 (Brumer–Stark). *Let $(K/k, S)$ be as above. Then, for all sets T satisfying the above conditions, we have*

$$\mathbf{BrSt}(K/k, S, T) : \quad \Theta_{S,T}(0) \in \mathrm{Ann}_{\mathbb{Z}[G]} \mathrm{CH}^1(K)_T^0.$$

Remark 6.3. *Let us fix $(K/k, S, T)$ as above. It is easily seen that if S contains at least two primes which split completely in K/k , then $\Theta_{S,T}(0) = 0$. If S contains exactly one prime which splits completely in K/k , then $\Theta_{S,T}(0) = 0$ unless k is an imaginary quadratic field, K/k is unramified everywhere and $S = S_\infty$. In the latter case, the proof of Conjecture $\mathbf{BrSt}(K/k, S, T)$ is an easy, albeit instructive exercise: Indeed, according to Corollary 4.3.3 (top change) in [24], one can assume that K is the Hilbert class-field of k . In that case, it is easily seen that*

$$\Theta_{S,T}(0) = \frac{\prod_{v \in T} (1 - \mathbf{N}v)}{w_k} \cdot \mathbf{N}_G,$$

where v runs through primes in k , $\mathbf{N}_G := \sum_{\sigma \in G} \sigma$ is the usual norm element in $\mathbb{Z}[G]$, and w_k is the cardinality of the group of roots of unity μ_k in k . Now, the desired annihilation result follows from the fact that ideals in k become principal in K (Hilbert’s capitulation theorem) and the exact sequence of abelian groups

$$1 \longrightarrow \mu_k \longrightarrow \Delta_{k,T} \longrightarrow C_{k,T} \longrightarrow C_k \longrightarrow 1.$$

(See exact sequence (10) and note that $U_k = \mu_k$ in this case.) We leave the remaining details to the interested reader. As a consequence, it suffices to study the Brumer–Stark Conjecture in the case where k is **totally real** and K is **totally imaginary**. (Otherwise, S will contain at least one infinite prime which splits completely in K/k and the above considerations settle the conjecture in that case.)

Now, Proposition 4.3.7 in [24] provides us with the following additional reduction.

Proposition 6.4. *Let $(K/k, S, T)$ be as above. Assume that k is totally real and K is totally imaginary. Let K^{CM} be the maximal CM subfield of K , let $G_{CM} := G(K^{CM}/k)$ and $\tilde{\Theta}_{S,T} := \Theta_{S,T, \bar{K}/k}$. Then, the following are equivalent, for all primes $p > 2$.*

- (1) $\Theta_{S,T}(0) \in \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathrm{CH}^1(K)_T^0 \otimes \mathbb{Z}_p)$.
- (2) $\tilde{\Theta}_{S,T}(0) \in \mathrm{Ann}_{\mathbb{Z}_p[G_{CM}]}(C_{K^{CM},T} \otimes \mathbb{Z}_p)^-.$

Proof. See Proposition 4.3.7 in [24]. The upper script “ $-$ ” in (2) refers to the action of the unique complex conjugation morphism in G^{CM} on $(C_{K^{CM},T} \otimes \mathbb{Z}_p)$. \square

As a consequence of the Equivariant Main Conjecture (Theorem 5.6), we can prove the following refinement of statement (2) in the Proposition above, under certain hypotheses (see below.) In what follows, S_p denotes the set of p -adic primes in k and $\mu_{K,p}$ denotes the Iwasawa μ -invariant associated to the number field K and the prime p .

Theorem 6.5. *Let $(K/k, S, T)$ be as above and let p be an odd prime. Assume that K is CM, k is totally real, $S_p \subseteq S$ and $\mu_{K,p} = 0$. Then, we have*

$$\overline{\text{BrSt}}(K/k, S, T, p) : \quad \Theta_{S,T}(0) \in \text{Fit}_{\mathbb{Z}_p[G]}((A_{K,T}^-)^\vee),$$

where $A_{K,T} := (C_{K,T} \otimes \mathbb{Z}_p)$ and the dual is endowed with the covariant G -action.

Proof. In what follows, all occurring \mathbb{Z}_p -module or Pontrjagin duals are endowed with the **covariant** action by the appropriate groups.

We let \mathcal{K} be the cyclotomic \mathbb{Z}_p -extension of K and $\mathcal{G} := \text{Gal}(\mathcal{K}/k)$. As usual, we let \mathcal{S} and \mathcal{T} denote the sets of finite primes in \mathcal{K} sitting above primes in S and T , respectively. Note that the hypotheses of Theorem 5.6 are satisfied by the data $(\mathcal{K}/k, S, T, p)$. Consequently, Corollary 5.24 implies that

$$(29) \quad \Theta_{S,T}^{(\infty)} \in \text{Fit}_{\mathbb{Z}_p[[\mathcal{G}]]}((T_p(\mathcal{M}_{S,\mathcal{T}}^\mathcal{K})^-)^*).$$

Now, recall that, with notations as in §2.2 and under the current hypotheses, the $\mathbb{Z}_p[[\mathcal{G}]]$ -module

$$\mathcal{A}_{\mathcal{K},\mathcal{T}}^- \simeq \varinjlim_n \mathcal{A}_{K_n,T_n}^-$$

is a torsion, divisible \mathbb{Z}_p -module of finite co-rank and that the transition maps in the inductive limit above are injective (see Lemma 2.9.) Consequently, since $\mathcal{A}_{\mathcal{K},\mathcal{T}}^-$ is finite, we can fix $n \in \mathbb{N}$, such that we have an inclusion of $\mathbb{Z}_p[[\mathcal{G}]]$ -modules

$$\mathcal{A}_{K,T}^- \subseteq \mathcal{A}_{\mathcal{K},\mathcal{T}}^-[p^n].$$

The inclusion above induces a natural surjection of $\mathbb{Z}_p[[\mathcal{G}]]$ -modules

$$(30) \quad \mathcal{A}_{\mathcal{K},\mathcal{T}}^-[p^n]^\vee \rightarrow (\mathcal{A}_{K,T}^-)^\vee.$$

Next, we need the following elementary result.

Lemma 6.6. *Let p be a prime, \mathcal{H} a profinite abelian group, and M a $\mathbb{Z}_p[[\mathcal{H}]]$ -module. Assume that M is \mathbb{Z}_p -torsion, divisible, of finite corank. Then there exist canonical isomorphisms of $\mathbb{Z}_p[[\mathcal{H}]]$ -modules*

$$M[p^m]^\vee \simeq T_p(M)^* \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m\mathbb{Z}, \quad \text{for all } m \in \mathbb{Z}_{\geq 1}.$$

Proof. Fix $m \in \mathbb{Z}_{\geq 1}$. There is a canonical isomorphism of $\mathbb{Z}_p[[\mathcal{H}]]$ -modules

$$T_p(M)^* \simeq M^\vee.$$

This induces a canonical isomorphism $T_p(M)^*/p^m \simeq M^\vee/p^m$. However, the exact functor $* \rightarrow \text{Hom}_{\mathbb{Z}_p}(*, \mathbb{Q}_p/\mathbb{Z}_p)$ applied to the exact sequence

$$0 \longrightarrow M[p^m] \longrightarrow M \xrightarrow{\times p^m} M \longrightarrow 0$$

induces an isomorphism $M^\vee/p^m \simeq M[p^m]^\vee$. This concludes the proof. \square

Consequently, we obtain the following surjective morphisms of $\mathbb{Z}_p[[\mathcal{G}]]$ -modules.

$$(31) \quad (T_p(\mathcal{M}_{S,T}^\mathcal{K})^-)^* \twoheadrightarrow T_p(\mathcal{A}_{K,T}^-)^* \twoheadrightarrow \mathcal{A}_{K,T}^-[p^n]^\vee \twoheadrightarrow (A_{K,T}^-)^\vee.$$

The first surjection above is obtained by taking \mathbb{Z}_p -duals in exact sequence (13). Note that $T_p(\mathcal{A}_{K,T}^-) = T_p(\mathcal{A}_{K,T})^-$. Also, note that the exact sequence (13) is split in the category of \mathbb{Z}_p -modules (because $\text{Div}_K(\mathcal{S} \setminus \mathcal{S}_p)^- \otimes \mathbb{Z}_p$ is \mathbb{Z}_p -free) and therefore it stays exact after taking \mathbb{Z}_p -duals. The second surjection is given by Lemma 6.6 applied to $M := \mathcal{A}_{K,T}^-$, $m := n$, $\mathcal{H} := \mathcal{G}$. Finally, the third surjection is (30) above.

Now, apply the first part of (44) in the Appendix for $R := \mathbb{Z}_p[[\mathcal{G}]]$, $M := (T_p(\mathcal{M}_{S,T}^\mathcal{K})^-)^*$ and $M' := (A_{K,T}^-)^\vee$, in combination with the surjections (31) as well as (29) above to conclude that we have

$$\Theta_{S,T}^{(\infty)} \in \text{Fit}_{\mathbb{Z}_p[[\mathcal{G}]]} T_p(\mathcal{M}_{S,T}^\mathcal{K})^- = \text{Fit}_{\mathbb{Z}_p[[\mathcal{G}]]} (T_p(\mathcal{M}_{S,T}^\mathcal{K})^-)^* \subseteq \text{Fit}_{\mathbb{Z}_p[[\mathcal{G}]]} ((A_{K,T}^-)^\vee).$$

Now, consider the projection $\pi : \mathbb{Z}_p[[\mathcal{G}]] \rightarrow \mathbb{Z}_p[G]$ given by Galois restriction. By the definition of $\Theta_{S,T}^{(\infty)}$, we have

$$\pi(\Theta_{S,T}^{(\infty)}) = \Theta_{S,T}(0), \quad A_{K,T}^- \simeq A_{K,T}^- \otimes_{\mathbb{Z}_p[[\mathcal{G}]]} \mathbb{Z}_p[G],$$

where the isomorphism is viewed in the category of $\mathbb{Z}_p[G]$ -modules. Consequently, (44) (see Appendix) applied for $M := A_{K,T}^-$ and $\rho := \pi$ implies that

$$\Theta_{S,T}(0) \in \text{Fit}_{\mathbb{Z}_p[G]} ((A_{K,T}^-)^\vee),$$

which concludes the proof of the Theorem. \square

Corollary 6.7. *Assume that $(K/k, S, T)$ satisfy the hypotheses in Conjecture 6.2. Let p be a prime. If k is totally real and K is totally imaginary, assume that p is odd, $S_p \subseteq S$ and $\mu_{K^{CM}, p} = 0$. Then, we have:*

$$\text{BrSt}(K/k, S, T, p) : \quad \Theta_{S,T}(0) \in \text{Ann}_{\mathbb{Z}_p[G]}(\text{CH}^1(K)_T^0 \otimes \mathbb{Z}_p).$$

Proof. Combine Theorem 6.5 with Proposition 6.4 and Remark 6.3 and note that

$$\text{Fit}_{\mathbb{Z}_p[G]}(M^\vee) \subseteq \text{Ann}_{\mathbb{Z}_p[G]}(M),$$

for any $\mathbb{Z}_p[G]$ -module M , if M^\vee is endowed with the covariant G -action. \square

Remark 6.8. *Some cases of the Brumer-Stark Conjecture over an arbitrary totally real number field were also settled by the first author in [11] with different methods and working under somewhat more restrictive hypotheses. See Theorem 10 in loc.cit.*

6.2. The Coates-Sinnott Conjecture. As usual, K/k is an abelian extension of number fields of Galois group G , and S is a finite set of primes in k , such that $S_{\text{ram}}(K/k) \cup S_\infty \subseteq S$.

If p is a prime number, we let $H_{\text{et}}^i(\mathcal{O}_{K,S}[1/p], \mathbb{Z}_p(n))$ denote the i -th étale cohomology group of the affine scheme $\text{Spec}(\mathcal{O}_{K,S}[1/p]) = \text{Spec}(\mathcal{O}_{K, S \cup S_p})$ with coefficients in the étale p -adic sheaf $\mathbb{Z}_p(n)$, for all $i \in \mathbb{Z}_{\geq 0}$ and all $n \in \mathbb{Z}_{\geq 2}$. Also, for every $m \in \mathbb{Z}_{\geq 0}$, we let $K_m(\mathcal{O}_{K,S})$ denote the m -th Quillen K-group of $\mathcal{O}_{K,S}$. For the definitions and properties of these étale cohomology groups and K-groups, the reader may consult Kolster's excellent survey article [17]. For a discussion closer in

spirit to the current section, the reader may consult [23] as well. In the present context, all these étale cohomology and K-theory groups come endowed with natural $\mathbb{Z}_p[G]$ -module, respectively $\mathbb{Z}[G]$ -module structures.

Quillen showed in [25] that $K_m(\mathcal{O}_{K,S})$ is a finitely generated abelian group, for all m . Borel showed in [1] that we have the following remarkable equalities

$$(32) \quad \text{rank}_{\mathbb{Z}} K_{2n-i}(\mathcal{O}_{K,S}) = \begin{cases} 0, & \text{if } i = 2; \\ \text{ord}_{s=(1-n)} \zeta_{K,S}(s), & \text{if } i = 1, \end{cases}$$

for all $n \geq 2$, where $\zeta_{K,S}$ is the S -incomplete zeta-function associated to K . For all primes $p > 2$, Soulé [30] and later Dwyer-Friedlander [8] constructed surjective $\mathbb{Z}_p[G]$ -linear p -adic Chern character morphisms

$$(33) \quad \text{ch}_{p,n}^i : K_{2n-i}(\mathcal{O}_{K,S}) \otimes \mathbb{Z}_p \rightarrow H_{et}^i(\mathcal{O}_{K,S}[1/p], \mathbb{Z}_p(n)), \quad \forall i = 1, 2, \quad \forall n \in \mathbb{Z}_{\geq 2}.$$

Soulé proved in [31] that these morphisms have finite kernels. Similar morphisms have been defined for $p = 2$ (see [8].) Their kernels and cokernels are finite but non trivial, in general.

Consequently, the group $H_{et}^2(\mathcal{O}_{K,S}[1/p], \mathbb{Z}_p(n))$ is finite while $H_{et}^1(\mathcal{O}_{K,S}[1/p], \mathbb{Z}_p(n))$ is finitely generated over \mathbb{Z}_p of \mathbb{Z}_p -rank equal to $\text{ord}_{s=(1-n)} \zeta_{K,S}(s)$, for all p and all $n \geq 2$. If $i \geq 3$, the group $H_{et}^i(\mathcal{O}_{K,S}[1/p], \mathbb{Z}_p(n))$ vanishes for $p > 2$ and is a finite, 2-primary group for $p = 2$ (see [17], §2.)

With notations as in §5, we have the following.

Lemma 6.9. *For all $n \geq 2$ and all primes p , the following hold.*

- (1) *We have a $\mathbb{Z}_p[G]$ -module isomorphism*

$$H_{et}^1(\mathcal{O}_{K,S}[1/p], \mathbb{Z}_p(n))_{\text{tors}} \simeq (\mathbb{Q}_p/\mathbb{Z}_p(n))^{G_K}.$$

- (2) *We have an equality of $\mathbb{Z}_p[G]$ -ideals*

$$\text{Ann}_{\mathbb{Z}_p[G]}(H_{et}^1(\mathcal{O}_{K,S}[1/p], \mathbb{Z}_p(n))_{\text{tors}}) = \langle \delta_{T,K/k}(1-n) \mid T \rangle,$$

where T runs through all the finite, non-empty sets of primes in k which are disjoint from $S \cap S_p$.

- (3) $\text{Ann}_{\mathbb{Z}_p[G]}(H_{et}^1(\mathcal{O}_{K,S}[1/p], \mathbb{Z}_p(n))_{\text{tors}}) \cdot \Theta_{S,K/k}(1-n) \subseteq \mathbb{Z}_p[G].$

Proof. For (1), see §2 in [17]. Note that in loc.cit., the author deals with the étale cohomology groups of $\text{Spec}(\mathcal{O}_K[1/p])$ rather than those of $\text{Spec}(\mathcal{O}_{K,S}[1/p])$. However, for all n and p as above, Soulé's localization sequence in étale cohomology (see [30]) establishes an isomorphism of $\mathbb{Z}_p[G]$ -modules

$$(34) \quad H_{et}^1(\mathcal{O}_K[1/p], \mathbb{Z}_p(n)) \simeq H_{et}^1(\mathcal{O}_{K,S}[1/p], \mathbb{Z}_p(n)).$$

Part (2) follows from part (1) and a well-known lemma of Coates (see Lemma 2.3 in [4] and its proof.) Also, see the proof of Corollary 5.3 above.

Part (3) is a direct consequence of part (2) and Corollary 5.3 above. □

Conjecture 6.10 (Coates-Sinnott, cohomological version). *For all $(K/k, S, p, n)$ as above, the following holds.*

$$\begin{aligned} \text{CS}(K/k, S, p, n) : \quad & \text{Ann}_{\mathbb{Z}_p[G]}(H_{et}^1(\mathcal{O}_{K,S}[1/p], \mathbb{Z}_p(n))_{\text{tors}}) \cdot \Theta_{S,K/k}(1-n) \subseteq \\ & \subseteq \text{Ann}_{\mathbb{Z}_p[G]}(H_{et}^2(\mathcal{O}_{K,S}[1/p], \mathbb{Z}_p(n))). \end{aligned}$$

With notations as above, let $e_n(K/k)$ be the idempotent in $\mathbb{Z}_p[G]$ given by

$$e_n(K/k) := \begin{cases} \prod_{v \in S_\infty(k)} \frac{1}{2}(1 + (-1)^n \sigma_v), & \text{if } k \text{ is totally real;} \\ 0, & \text{otherwise,} \end{cases}$$

where σ_v is the generator of the decomposition group G_v , for all $v \in S_\infty(k)$. The main goal of this section is a proof of the following refinement of the conjecture above, under the expected hypotheses.

Theorem 6.11. *Assume that $(K/k, S, p, n)$ are as above. If k is totally real, assume that $p > 2$ and $\mu_{K(\mu_p)^{CM}, p} = 0$, where $K(\mu_p)^{CM}$ is the maximal CM-subfield of $K(\mu_p)$. Then, the following holds.*

$$\begin{aligned} \overline{\text{CS}}(K/k, S, p, n) : \quad \text{Ann}_{\mathbb{Z}_p[G]}(H_{\text{et}}^1(\mathcal{O}_{K,S}[1/p], \mathbb{Z}_p(n))_{\text{tors}}) \cdot \Theta_{S, K/k}(1-n) = \\ = e_n(K/k) \cdot \text{Fit}_{\mathbb{Z}_p[G]}(H_{\text{et}}^2(\mathcal{O}_{K,S}[1/p], \mathbb{Z}_p(n))). \end{aligned}$$

Proof. We begin by making a few useful reduction steps.

Lemma 6.12. *It suffices to prove statement $\overline{\text{CS}}(K/k, S, p, n)$ under the assumption that $S_p \subseteq S$, where S_p denotes the set of p -adic primes in k .*

Proof. This is an immediate consequence of (34) and the obvious equality in $\mathbb{Z}_p[G]$

$$\Theta_{S \cup S_p, K/k}(1-n) = \Theta_{S, K/k}(1-n) \cdot \prod_{v \in S_p \setminus S} (1 - \sigma_v^{-1} \cdot (\mathbf{N}v)^n),$$

for all n and p as above. Indeed, for all $v \in S_p \setminus S$, the element $(1 - \sigma_v^{-1} \cdot (\mathbf{N}v)^n)$ is a divisor of $(1 - p^m)$ in $\mathbb{Z}_p[G]$, for a large m (say, m such that $p^m = |\kappa(w)|^n$, for w prime in K dividing v) and it is therefore a unit in $\mathbb{Z}_p[G]$. Therefore, the elements $\Theta_{S \cup S_p, K/k}(1-n)$ and $\Theta_{S, K/k}(1-n)$ differ by a unit in $\mathbb{Z}_p[G]$. \square

Lemma 6.13. *It suffices to prove statement $\overline{\text{CS}}(K/k, S, p, n)$ under the assumption that k is a totally real number field.*

Proof. Indeed, the functional equation satisfied by the S -incomplete L -function associated to a character $\chi \in \widehat{G}(\mathbb{C})$ implies the following formula for the order of vanishing at $s = (1-n)$, for all $n \in \mathbb{Z}_{\geq 2}$.

$$(35) \quad \text{ord}_{s=(1-n)} L_S(\chi, s) = \begin{cases} r_2(k) + a(\chi)^+, & \text{if } n \text{ is odd;} \\ r_2(k) + a(\chi)^-, & \text{if } n \text{ is even,} \end{cases}$$

where $r_2(k)$ denotes the number of complex infinite places in k ,

$$a(\chi)^+ = \text{card}\{v \in S_\infty(k) \mid \chi|_{G_v} = \mathbf{1}_{G_v}\}, \quad a(\chi)^- = \text{card}\{v \in S_\infty(k) \mid \chi|_{G_v} \neq \mathbf{1}_{G_v}\}.$$

Note that $a(\chi)^+ + a(\chi)^- = r_1(k)$, the number of real infinite places in k . The above formula shows that if $r_2(k) > 0$ (i.e. if k is not totally real), then we have $\Theta_{S, K/k}(1-n) = 0$, for all $n \in \mathbb{Z}_{\geq 2}$. This concludes the proof of the Lemma. \square

Lemma 6.14. *Assume that \tilde{K}/k is an abelian extension of number fields, of Galois group \tilde{G} , with $K \subseteq \tilde{K}$ and $S_{\text{ram}}(\tilde{K}/k) \subseteq S$. Then, for all $p > 2$ and n as above,*

$$\overline{\text{CS}}(\tilde{K}/k, S, p, n) \Rightarrow \overline{\text{CS}}(K/k, S, p, n).$$

Proof. Let us fix $p > 2$ and n as above. In order to simplify notations, we let $H_K^i := H_{et}^i(\mathcal{O}_{K,S}[1/p], \mathbb{Z}_p(n))$ and similarly for $H_{\tilde{K}}^i$, for all $i = 1, 2$. Let $H := G(\tilde{K}/K)$. Galois restriction induces the usual \mathbb{Z}_p -algebra morphism $\pi : \mathbb{Z}_p[\tilde{G}] \rightarrow \mathbb{Z}_p[G]$, whose kernel is the relative augmentation ideal I_H associated to H in $\mathbb{Z}_p[\tilde{G}]$. Proposition 2.10 in [17] gives a canonical $\mathbb{Z}_p[G]$ -module isomorphism

$$H_K^2 \simeq (H_{\tilde{K}}^2)_H, \quad H_K^1 \simeq (H_{\tilde{K}}^1)^H$$

where $M^H := M[I_H]$ and $M_H := M/I_H M \simeq M \otimes_{\mathbb{Z}_p[\tilde{G}]} \mathbb{Z}_p[G]$ denote the modules of H -invariants and H -coinvariants, respectively, associated to any $\mathbb{Z}_p[\tilde{G}]$ -module M . The isomorphism above combined with the second equality in (44) of the Appendix (applied for $\rho := \pi$, $R := \mathbb{Z}_p[\tilde{G}]$, $R' := \mathbb{Z}_p[G]$, and $M := H_K^2$) gives

$$(36) \quad \pi(\text{Fit}_{\mathbb{Z}_p[\tilde{G}]}(H_K^2)) = \text{Fit}_{\mathbb{Z}_p[G]}(H_K^2).$$

The above equality combined with the obvious equalities $\pi(e_n(\tilde{K}/k)) = e_n(K/k)$, $\pi(\Theta_{S, \tilde{K}/k}(1-n)) = \Theta_{S, K/k}(1-n)$ and

$$(37) \quad \pi(\text{Ann}_{\mathbb{Z}_p[\tilde{G}]}(H_K^1)_{\text{tors}}) = \text{Ann}_{\mathbb{Z}_p[G]}(H_K^1)_{\text{tors}}$$

concludes the proof of the Lemma. (Apply Lemma 6.9(2) for the last equality.) \square

Lemma 6.15. *With notations as in the previous Lemma, assume that k is totally real, \tilde{K} is totally imaginary and $K := \tilde{K}^{CM}$ is the maximal CM subfield of \tilde{K} . Then, for all $p > 2$ and n as above, we have*

$$\overline{\text{CS}}(\tilde{K}/k, S, p, n) \Leftrightarrow \overline{\text{CS}}(K/k, S, p, n).$$

Proof. Let us fix $p > 2$ and n as above. We use the notations in the proof of the previous Lemma. Note that H is the 2-group generated by the set

$$\{\sigma_v \cdot \sigma_{v'} \mid v, v' \in S_{\infty}(k), v \neq v'\}.$$

This shows that, for all $\chi \in \hat{G}(\mathbb{C})$, we have

$$\chi|_H \neq \mathbf{1}_H \Rightarrow L_S(\chi, 1-n) = 0, \quad \chi(e_n(\tilde{K}/k)) = 0.$$

Indeed, if $\chi|_H \neq \mathbf{1}_H$, then there exist $v, v' \in S_{\infty}(k)$, with $v \neq v'$, such that $\chi(\sigma_v) = 1$ and $\chi(\sigma_{v'}) \neq 1$. On one hand, this shows that $\chi(e_n(\tilde{K}/k)) = 0$. On the other hand, it shows that $a(\chi)^+ \geq 1$ and $a(\chi)^- \geq 1$. Now, (35) implies that $L_S(\chi, 1-n) = 0$, for all $n \geq 2$. Consequently, if $e_H := |H|^{-1} \cdot \sum_{h \in H} h$ is the idempotent element associated to H in $\mathbb{Z}_p[\tilde{G}]$ (note that $p \nmid |H|$), we have

$$\Theta_{S, \tilde{K}/k}(1-n) \in e_H \mathbb{Q}_p[\tilde{G}], \quad e_n(\tilde{K}/k) \in e_H \mathbb{Z}_p[\tilde{G}].$$

Now, note that since $\mathbb{Z}_p[\tilde{G}] = e_H \mathbb{Z}_p[\tilde{G}] \oplus (1-e_H) \mathbb{Z}_p[\tilde{G}]$ and $I_H = (1-e_H) \mathbb{Z}_p[\tilde{G}]$, the map π establishes a ring isomorphism $\pi : e_H \mathbb{Z}_p[\tilde{G}] \simeq \mathbb{Z}_p[G]$. Consequently, (36) and (37) imply that π establishes an isomorphism at the level of $e_H \mathbb{Z}_p[\tilde{G}]$ -ideals

$$\pi : e_H \text{Fit}_{\mathbb{Z}_p[\tilde{G}]}(H_K^2) \simeq \text{Fit}_{\mathbb{Z}_p[G]}(H_K^2), \quad \pi : e_H \text{Ann}_{\mathbb{Z}_p[\tilde{G}]}(H_K^1)_{\text{tors}} \simeq \text{Ann}_{\mathbb{Z}_p[G]}(H_K^1)_{\text{tors}}.$$

Now, the equalities $\pi(e_n(\tilde{K}/k)) = e_n(K/k)$ and $\pi(\Theta_{S, \tilde{K}/k}(1-n)) = \Theta_{S, K/k}(1-n)$ conclude the proof of the Lemma. \square

Now, we are ready to return to the proof of Theorem 6.11. We fix $(K/k, S, p)$ as in the theorem. According to the previous four Lemmas, we may assume that k is totally real (otherwise, the statement is trivially true, according to Lemma 6.13), $S_p \subseteq S$ (see Lemma 6.12), K is CM and $\mu_p \subseteq K$ (otherwise, we replace K by $K(\mu_p)^{CM}$ and apply Lemmas 6.15 and 6.14, respectively.) Under these hypotheses, we assume in addition that $p > 2$ and $\mu_{K,p} = 0$.

As usual, we let \mathcal{K} denote the cyclotomic \mathbb{Z}_p -extension of K , $\mathcal{G} := G(\mathcal{K}/k)$ and $\Gamma := G(\mathcal{K}/K)$. We fix a finite, nonempty set T set of primes in K , such that $S \cap T = \emptyset$. Also, we let \mathcal{S} and \mathcal{T} denote the sets of finite primes in \mathcal{K} sitting above primes in S and T , respectively. For simplicity, we let $\Theta_S := \Theta_{S, \mathcal{K}/k}$, $\Theta_{S,T} := \Theta_{S,T, \mathcal{K}/k}$ and $\delta_T := \delta_{T, \mathcal{K}/k}$ and resume using all the notations introduced in §5. In particular, $j \in \mathcal{G}$ denotes the unique complex conjugation automorphism of \mathcal{K} (identified as usual with the complex conjugation automorphism $j \in G$ of K .) Note that we have

$$e_n := e_n(K/k) = \frac{1}{2}(1 + (-1)^n j), \quad \forall n \in \mathbb{Z}.$$

We view e_n either as an element of $\mathbb{Z}_p[[\mathcal{G}]]$ or $\mathbb{Z}_p[G]$, for all $n \in \mathbb{Z}$. Observe that if M is a $\mathbb{Z}_p[[\mathcal{G}]]$ -module and $n \in \mathbb{Z}$, then $e_n M = M^+$, if n is even and $e_n M = M^-$, if n is odd. Also, note that as a consequence of (35) we have

$$(38) \quad \Theta_S(1-n) = e_n \cdot \Theta_S(1-n), \quad \Theta_{S,T}(1-n) = e_n \cdot \Theta_{S,T}(1-n), \quad \forall n \in \mathbb{Z}_{\geq 2}.$$

In what follows, all the occurring \mathbb{Z}_p -module duals or Pontrjagin duals are endowed with the **contravariant** actions by the appropriate groups, unless stated otherwise. We will need the following elementary result.

Lemma 6.16. *Let M be a $\mathbb{Z}_p[[\mathcal{G}]]$ -module. Assume that M is \mathbb{Z}_p -free of finite rank and that M_Γ is finite. Then,*

- (1) $M^\Gamma = 0$.
- (2) $(M_\Gamma)^\vee \simeq (M^*)_\Gamma$, as $\mathbb{Z}_p[G]$ -modules.
- (3) If $\text{pd}_{\mathbb{Z}_p[[\mathcal{G}]]} M \leq 1$, then $\text{pd}_{\mathbb{Z}_p[G]} M_\Gamma \leq 1$.

Proof. Part (1) is immediate. For part (2), see Lemma 5.18 in [13]. Part (3) is an immediate consequence of $\mathbb{Z}_p[[\mathcal{G}]]^\Gamma = 0$ and $\mathbb{Z}_p[[\mathcal{G}]]_\Gamma \simeq \mathbb{Z}_p[G]$. \square

Proposition 6.17. *Let $n \in \mathbb{Z}_{\geq 2}$, $p > 2$, and $H_K^i := H_{et}^i(\mathcal{O}_{K,S}, \mathbb{Z}_p(n))$, for all $i = 1, 2$. Then, we have $\mathbb{Z}_p[G]$ -module isomorphisms*

- (1) $e_n \cdot H_K^2 \simeq (\mathfrak{X}_S^+(-n)_\Gamma)^\vee \simeq (\mathfrak{X}_S^+(-n)^*)_\Gamma$;
- (2) $e_n \cdot H_K^1 = e_n \cdot (H_K^1)_{\text{tors}} \simeq (H_K^1)_{\text{tors}} \simeq \mathbb{Z}_p(n)_\Gamma$,

where M^Γ and M_Γ denote the $\mathbb{Z}_p[G]$ -modules of Γ -invariants and Γ -coinvariants, respectively, for all $\mathbb{Z}_p[[\mathcal{G}]]$ -modules M .

Proof. Let $H_{K^+}^i := H_{et}^i(\mathcal{O}_{K^+,S}, \mathbb{Z}_p(n))$, for all $i = 1, 2$, where $K^+ := K^{j=1}$, as usual. Then, Proposition 2.9 in [17] shows that there is a canonical $\mathbb{Z}_p[G]$ -module isomorphism $H_{K^+}^1 \simeq (H_K^1)^+$. Consequently, (33) and (32) together with (35) give

$$\text{rank}_{\mathbb{Z}_p}(e_n \cdot H_K^1) = \begin{cases} r_2(K) - r_1(K^+), & \text{if } n \text{ is odd;} \\ r_2(K^+), & \text{if } n \text{ is even} \end{cases} = 0.$$

Consequently, $e_n H_K^1$ is a finite group. On one hand, this implies the equality in part (2) of the Proposition. On the other hand, via the spectral sequence argument on pp. 237–238 of [17] this leads to isomorphisms of $\mathbb{Z}_p[G]$ -modules

$$e_n \cdot H_K^2 \simeq e_n \cdot H_{et}^1(\mathcal{O}_{K,S}, \mathbb{Q}_p/\mathbb{Z}_p(n)) \simeq e_n \cdot H^1(G_{\mathcal{K},S}^{(p)}, \mathbb{Q}_p/\mathbb{Z}_p(n))^\Gamma,$$

where $G_{\mathcal{K},S}^{(p)}$ is the Galois group of the maximal pro- p extension of \mathcal{K} which is unramified away from \mathcal{S} and the right-most cohomology group is a Galois cohomology group. Now, since $\mu_{p^\infty} \subseteq \mathcal{K}$ and therefore $G_{\mathcal{K},S}^{(p)}$ acts trivially on $\mathbb{Q}_p/\mathbb{Z}_p(n)$ and since $\mathfrak{X}_{\mathcal{S}}$ is the maximal abelian quotient of $G_{\mathcal{K},S}^{(p)}$, we have the following isomorphisms of $\mathbb{Z}_p[G]$ -modules.

$$\begin{aligned} H^1(G_{\mathcal{K},S}^{(p)}, \mathbb{Q}_p/\mathbb{Z}_p(n))^\Gamma &\simeq \operatorname{Hom}_{\mathbb{Z}_p}(G_{\mathcal{K},S}^{(p)}, \mathbb{Q}_p/\mathbb{Z}_p(n))^\Gamma \\ &\simeq \operatorname{Hom}_{\mathbb{Z}_p}(\mathfrak{X}_{\mathcal{S}}, \mathbb{Q}_p/\mathbb{Z}_p(n))^\Gamma \simeq (\mathfrak{X}_{\mathcal{S}}(-n)_\Gamma)^\vee. \end{aligned}$$

Now, the first isomorphism in part (1) of the Proposition follows from the last two displayed isomorphisms and the obvious equality $e_n \cdot \mathfrak{X}_{\mathcal{S}}(-n) = \mathfrak{X}_{\mathcal{S}}^+(-n)$.

The second isomorphism in part (1) follows from Lemma 6.16(2) applied to $M := \mathfrak{X}_{\mathcal{S}}^+(-n)$. Note that the finiteness of H_K^2 and the first isomorphism in part (1) of the Proposition imply that $\mathfrak{X}_{\mathcal{S}}^+(-n)_\Gamma$ is finite. Also, $\mathfrak{X}_{\mathcal{S}}^+(-n)$ is finitely generated over \mathbb{Z}_p (a theorem of Iwasawa) and therefore \mathbb{Z}_p -free of finite rank (a consequence of $\mu_{K,p} = 0$.) This concludes the proof of part (1) of the Proposition.

The isomorphism in part (2) of the Proposition is an immediate consequence of the isomorphism $(H_K^1)_{\text{tors}} \simeq (\mathbb{Q}_p/\mathbb{Z}_p(n))^\Gamma$ (see Lemma 6.9(1) and recall that $\mu_{p^\infty} \subseteq \mathcal{K}$) and the obvious equality $\mathbb{Q}_p(n)_\Gamma = \mathbb{Q}_p(n)_\Gamma = 0$. \square

Note that the hypotheses of Lemma 3.9 and Theorem 5.6 are satisfied by the data $(\mathcal{K}/k, \mathcal{S}, \mathcal{T}, p)$. Lemma 3.9 and exact sequence (15) tensored with $\mathbb{Z}_p(n-1)$ lead to the following four term exact sequence of $\mathbb{Z}_p[[G]]$ -modules.

$$(39) \quad 0 \rightarrow \mathbb{Z}_p(n) \rightarrow T_p(\Delta_{\mathcal{K},\mathcal{T}})^-(n-1) \rightarrow T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^\mathcal{K})^-(n-1) \rightarrow \mathfrak{X}_{\mathcal{S}}^+(-n)^* \rightarrow 0.$$

We intend to apply Proposition 7.3 in the Appendix to the sequence of Γ -coinvariants associated to the exact sequence above.

First, let us note that each of the four modules in the exact sequence above is \mathbb{Z}_p -free. Most importantly, the module of Γ -coinvariants associated to each of these is finite. Indeed, $\mathbb{Z}_p(n)_\Gamma$ and $(\mathfrak{X}_{\mathcal{S}}^+(-n)^*)_\Gamma$ are finite, due to Proposition 6.17. On the other hand, Lemma 5.22(1) and Remark 5.23 combined with Lemma 7.4(1) in the Appendix imply that we have a $\mathbb{Z}_p[G]$ -module isomorphism

$$T_p(\Delta_{\mathcal{K},\mathcal{T}})(n-1)_\Gamma \simeq \bigoplus_{v \in T} \mathbb{Z}_p[G]/(1 - \sigma_v^{-1} \cdot N_v^n).$$

Since $n \geq 2$, the element $(1 - \sigma_v^{-1} \cdot N_v^n)$ is not a zero-divisor in $\mathbb{Z}_p[G]$, for all $v \in T$. Consequently, the above isomorphism implies that $T_p(\Delta_{\mathcal{K},\mathcal{T}})(n-1)_\Gamma$ is finite. Therefore, its direct summand $T_p(\Delta_{\mathcal{K},\mathcal{T}})^-(n-1)_\Gamma$ is indeed finite. Now, the finiteness of $T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^\mathcal{K})^-(n-1)_\Gamma$ follows from the exact sequence

$$T_p(\Delta_{\mathcal{K},\mathcal{T}})^-(n-1)_\Gamma \rightarrow T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^\mathcal{K})^-(n-1)_\Gamma \rightarrow (\mathfrak{X}_{\mathcal{S}}^+(-n)^*)_\Gamma \rightarrow 0.$$

Now, Lemma 6.16 implies that the Γ -invariants of the four modules in (39) are trivial. Consequently, we obtain the following exact sequence of finite $\mathbb{Z}_p[G]$ -modules at the level of Γ -coinvariants

$$(40) \quad 0 \rightarrow e_n \cdot H_K^1 \rightarrow T_p(\Delta_{\mathcal{K}, \mathcal{T}})^-(n-1)_\Gamma \rightarrow T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^\mathcal{K})^-(n-1)_\Gamma \rightarrow e_n \cdot H_K^2 \rightarrow 0.$$

Above, we have used Proposition 6.17 to identify the Γ -coinvariants of the end terms of (39) with $e_n \cdot H_K^1$ and $e_n \cdot H_K^2$, respectively.

Since $\text{pd}_{\mathbb{Z}_p[[\mathcal{G}]]} T_p(\Delta_{\mathcal{K}, \mathcal{T}})^- \leq 1$ and $\text{pd}_{\mathbb{Z}_p[[\mathcal{G}]]} T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^\mathcal{K})^- \leq 1$ (see Lemma 5.22(3) and Remark 5.23 for the first and Theorem 4.6(2) for the second), we have

$$\text{pd}_{\mathbb{Z}_p[G]} (T_p(\Delta_{\mathcal{K}, \mathcal{T}})^-(n-1)_\Gamma) \leq 1, \quad \text{pd}_{\mathbb{Z}_p[G]} (T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^\mathcal{K})^-(n-1)_\Gamma) \leq 1,$$

as a consequence of 7.4(3) and Lemma 6.16(3). Consequently, we may apply Proposition 7.3 in the Appendix to the exact sequence (40). This way, we obtain

$$(41) \quad \begin{aligned} & \text{Fit}_{\mathbb{Z}_p[G]}(e_n H_K^1)^\vee \cdot \text{Fit}_{\mathbb{Z}_p[G]}(T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^\mathcal{K})^-(n-1)_\Gamma) = \\ & = \text{Fit}_{\mathbb{Z}_p[G]}(e_n H_K^2) \cdot \text{Fit}_{\mathbb{Z}_p[G]}(T_p(\Delta_{\mathcal{K}, \mathcal{T}})^-(n-1)_\Gamma). \end{aligned}$$

where the dual is endowed with the covariant G -action. Now, let us note that for any $\mathbb{Z}_p[[\mathcal{G}]]$ -module M and $\mathbb{Z}_p[G]$ -module N , we have

$$e_n \cdot M(n-1) = M^-(n-1), \quad e_n \cdot \text{Fit}_{\mathbb{Z}_p[G]}(N) = \text{Fit}_{e_n \mathbb{Z}_p[G]}(e_n \cdot N).$$

Consequently, if we combine Lemma 7.4(2) in the Appendix with Corollary 5.24 and Lemma 5.22(2), respectively, we obtain

$$\begin{aligned} e_n \cdot \text{Fit}_{\mathbb{Z}_p[G]}(T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^\mathcal{K})^-(n-1)_\Gamma) &= e_n \cdot (\pi \circ t_{1-n}(\Theta_{S, T}^{(\infty)})) = (\Theta_{S, T}(1-n)), \\ e_n \cdot \text{Fit}_{\mathbb{Z}_p[G]}(T_p(\Delta_{\mathcal{K}, \mathcal{T}})^-(n-1)_\Gamma) &= e_n \cdot (\pi \circ t_{1-n}(\delta_T^{(\infty)})) = e_n \cdot (\delta_T(1-n)), \end{aligned}$$

where $\pi : \mathbb{Z}_p[[\mathcal{G}]] \rightarrow \mathbb{Z}_p[G]$ is the usual projection. Note that above we have used the second equality in (38). Consequently, (41) implies that

$$\text{Fit}_{\mathbb{Z}_p[G]}(e_n H_K^1)^\vee \cdot \Theta_{S, T}(1-n) = e_n \text{Fit}_{\mathbb{Z}_p[G]}(H_K^2) \cdot (\delta_T(1-n)).$$

However, since $\Theta_{S, T}(1-n) = \delta_T(1-n) \cdot \Theta_S(1-n)$ and $\delta_T(1-n)$ is not a zero-divisor in $\mathbb{Z}_p[G]$ (an easy exercise!), the last equality implies

$$\text{Fit}_{\mathbb{Z}_p[G]}((H_K^1)_{\text{tors}})^\vee \cdot \Theta_S(1-n) = e_n \text{Fit}_{\mathbb{Z}_p[G]}(H_K^2).$$

Now, since $(H_K^1)_{\text{tors}}$ is a cyclic module (see Lemma 6.16(2)) and the dual is endowed with the covariant G -action, we have equalities

$$\text{Fit}_{\mathbb{Z}_p[G]}((H_K^1)_{\text{tors}})^\vee = \text{Ann}_{\mathbb{Z}_p[G]}((H_K^1)_{\text{tors}})^\vee = \text{Ann}_{\mathbb{Z}_p[G]}((H_K^1)_{\text{tors}}).$$

When combined with the last displayed equality, this concludes the proof of Theorem 6.11 (our refinement of the cohomological Coates-Sinnott Conjecture.) \square

Remark 6.18. *Results somewhat weaker than our Theorem 6.11 were obtained with different methods in [2], Corollary 2 (which imposes restrictions upon K/k and p) and [22], Théorème 4.3 (which imposes restrictions upon K/k and n .) In both cases, the vanishing of the appropriate Iwasawa μ -invariant is assumed.*

Finally, we would like to mention that the well known Quillen-Lichtenbaum Conjecture states that the Chern character maps $\text{ch}_{p,n}^i$ (see (33) above) are isomorphisms, for all $p > 2$ and $n \geq 2$. On the other hand, it is known that the Quillen-Lichtenbaum Conjecture is a consequence of the Bloch-Kato Conjecture for

finitely generated fields (e.g., see Theorem 2.7 in [17].) To our knowledge, recent work of Rost and Voevodsky has lead to a proof of the Bloch-Kato Conjecture.

The following is an immediate consequence of Theorem 6.11.

Corollary 6.19 (a refined K-theoretic Coates–Sinnott Conjecture). *Let K/k be an abelian extension of number fields of Galois group G . Let S be a finite set of primes in k , such that $S_\infty(k) \cup S_{\text{ram}}(K/k) \subseteq S$. If k is totally real, assume that $\mu_{K(\mu_p)^{CM},p} = 0$, for all primes $p > 2$. Also, assume that the Quillen–Lichtenbaum Conjecture holds. Then, for all $n \in \mathbb{Z}_{\geq 2}$, we have the following equality of $\mathbb{Z}[1/2][G]$ -ideals.*

$$\begin{aligned} \mathbb{Z}[1/2] \operatorname{Ann}_{\mathbb{Z}[G]}(K_{2n-1}(\mathcal{O}_{K,S})_{\text{tors}}) \cdot \Theta_S(1-n) = \\ = e_n(K/k) \cdot \mathbb{Z}[1/2] \operatorname{Fit}_{\mathbb{Z}[G]}(K_{2n-2}(\mathcal{O}_{K,S})). \end{aligned}$$

As the reader will notice right away, the K-theoretic statement above is closer in spirit to the conjecture originally formulated by Coates and Sinnott in [5]. The difference is the presence of a Fitting ideal rather than an annihilator on the right-hand side and an equality rather than an inclusion of ideals. These differences justify the use of the term “refined” above.

7. APPENDIX: ALGEBRAIC INGREDIENTS

7.1. Determinants and Fitting Ideals. Let R be a commutative ring with 1, P a finitely generated, projective R -module and $f \in \operatorname{End}_R(P)$. Then, the determinant $\det_R(f \mid P)$ of f acting on P is defined as follows. We take a finitely generated R -module Q , such that $P \oplus Q$ is a (finitely generated) free R -module, then we let $f \oplus \mathbf{1}_Q \in \operatorname{End}_R(P \oplus Q)$, where $\mathbf{1}_Q$ is the identity of Q , and define

$$\det_R(f \mid P) := \det_R(f \oplus \mathbf{1}_Q \mid P \oplus Q).$$

It is easy to check (use Schanuel’s Lemma !) that the definition above does not depend on Q . Now, one can use the same strategy to define the characteristic polynomial $\det_R(X - f \mid P) \in R[X]$ of variable X . Indeed, $P \otimes_R R[X]$ is a finitely generated, projective $R[X]$ -module. One defines

$$\det_R(X - f \mid P) := \det_{R[X]}(\operatorname{id}_P \otimes X - f \otimes 1 \mid P \otimes_R R[X]).$$

For any P , R and f as above and any R -algebra R' , we have base-change equalities

$$(42) \quad \det_R(f \mid P) = \det_{R'}(f \otimes \mathbf{1}_{R'} \mid P \otimes_R R'),$$

$$\det_R(X - f \mid P) = \det_{R'}(X - (f \otimes \mathbf{1}_{R'}) \mid P \otimes_R R').$$

Remark 7.1. *It is easily checked that the polynomial $F(X) := \det_R(X - f \mid P)$ in $R[X]$ defined above is always a monic polynomial.*

Now, for any R as above and any finitely presented R -module M , the first Fitting invariant (ideal) $\operatorname{Fit}_R(M)$ of M over R is defined as follows. First, one considers a finite presentation of M

$$R^n \xrightarrow{\phi} R^m \longrightarrow M \longrightarrow 0.$$

By definition, the Fitting ideal $\text{Fit}_R(M)$ is the ideal in R generated by the determinants of all the $m \times m$ minors of the matrix A_ϕ associated to ϕ with respect to R -bases of R^n and R^m . It is well-known that the definition does not depend on the chosen presentation or bases, and

$$(43) \quad \text{Ann}_R(M)^m \subseteq \text{Fit}_R(M) \subseteq \text{Ann}_R(M).$$

Two elementary properties of Fitting ideals which are used throughout the paper state that if $M \twoheadrightarrow M'$ is a surjective morphism of finitely presented R -modules and $\rho : R \rightarrow R'$ is a morphism of commutative rings with 1, then one has

$$(44) \quad \text{Fit}_R(M) \subseteq \text{Fit}_R(M'), \quad \text{Fit}_{R'}(M \otimes_R R') = \rho(\text{Fit}_R(M))R'.$$

For more details on general properties of Fitting ideals, the reader can consult the Appendix of [18].

In what follows, if R is a commutative topological ring and Γ is a profinite group, then the profinite group algebra

$$R[[\Gamma]] := \varprojlim_{\mathfrak{H}} R[\Gamma/\mathfrak{H}],$$

where Γ/\mathfrak{H} are all the finite quotients of Γ by (open and) closed subgroups \mathfrak{H} , is viewed as a topological R -algebra endowed with the usual projective limit topology.

Below, by a semi-local ring R we mean a direct sum of finitely many local rings. Examples of such rings include $\mathcal{O}[G]$ and $\mathcal{O}[G]^\pm := \mathcal{O}[G]/(1 \mp j)$, where G is a finite, abelian group, \mathcal{O} is a finite integral extension of \mathbb{Z}_p , for some odd prime p and j is an element of order 2 in G .

Proposition 7.2. *Let R be a commutative, semi-local, compact topological ring and Γ a pro-cyclic group of topological generator g . Let M be a topological $R[[\Gamma]]$ -module, which is projective and finitely generated as an R -module. Let*

$$F(X) := \det_R(X - \mathbf{m}_g \mid M),$$

where \mathbf{m}_g is the $R[[\Gamma]]$ -module automorphism of M given by multiplication by g . Then, the following hold.

- (1) *M is finitely presented as an $R[[\Gamma]]$ -module. Also, if we let $F(g)$ be the image of $F(X)$ via the R -algebra morphism $R[X] \rightarrow R[[\Gamma]]$ sending X to g , we have an equality of $R[[\Gamma]]$ -ideals*

$$\text{Fit}_{R[[\Gamma]]}(M) = (F(g)).$$

- (2) *Let $M_R^* := \text{Hom}_R(M, R)$, viewed as a topological $R[[\Gamma]]$ -module with the covariant Γ -action, given by $\sigma \cdot f(x) := f(\sigma \cdot x)$, for all $f \in M_R^*$, $\sigma \in \Gamma$ and $x \in M$. Then, we have*

$$\text{Fit}_{R[[\Gamma]]}(M) = \text{Fit}_{R[[\Gamma]]}(M_R^*).$$

- (3) *Assume that $R = \mathbb{Z}_p[G]$, where G is a finite, abelian group and p is a prime number. Let $M^* := \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$, viewed as an $R[[\Gamma]] \simeq \mathbb{Z}_p[[G \times \Gamma]]$ -module with the covariant $G \times \Gamma$ -action. Then, we have*

$$\text{Fit}_{R[[\Gamma]]}(M^*) = \text{Fit}_{R[[\Gamma]]}(M).$$

Proof. See Proposition 4.1 and Corollary 4.2 in [13]. □

Proposition 7.3. *Let $R := \mathbb{Z}_p[G]$, for some finite abelian group G and prime number p . Assume that we have an exact sequence of finite R -modules*

$$0 \longrightarrow A \longrightarrow P \longrightarrow P' \longrightarrow A' \longrightarrow 0.$$

Further, assume that $\mathrm{pd}_{\mathbb{Z}_p[G]} P \leq 1$ and $\mathrm{pd}_{\mathbb{Z}_p[G]} P' \leq 1$. Then, we have

$$\mathrm{Fit}_R(A^\vee) \cdot \mathrm{Fit}_R(P') = \mathrm{Fit}_R(A) \cdot \mathrm{Fit}_R(P),$$

where the dual $A^\vee := \mathrm{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$ is endowed with the covariant G -action.

Proof. See Lemma 5, p.179 of [2]. In loc.cit. this result is proved for general finitely generated \mathbb{Z}_p -algebras R which are \mathbb{Z}_p -free and relatively Gorenstein over \mathbb{Z}_p . Also, note that $\mathrm{pd}_{\mathbb{Z}_p[G]} P = 0$ if and only if $P = 0$ (and similarly for P'), in which case the equality above is immediate. \square

7.2. Twisting. In what follows, we fix an odd prime p , a field k of characteristic different from p and a Galois extension \mathcal{K}/k . We let $\mathcal{G} := \mathrm{Gal}(\mathcal{K}/k)$ and assume that the group of p -power roots of unity μ_{p^∞} is contained in \mathcal{K} . As usual, we denote by $c_p : \mathcal{G} \rightarrow \mathbb{Z}_p^\times = \mathrm{Aut}(\mu_{p^\infty})$ the p -cyclotomic character of \mathcal{G} and decompose $c_p := \omega_p \cdot \kappa_p$ in its tame (Teichmüller) and wild components, $\omega_p : \mathcal{G} \rightarrow \mu_{p-1}$ and $\kappa_p : \mathcal{G} \rightarrow (1+p\mathbb{Z}_p)$, respectively. For simplicity, we let $c := c_p$, $\omega := \omega_p$ and $\kappa := \kappa_p$.

Let \mathcal{O} be a finite, integral extension of \mathbb{Z}_p . We let $Q(\mathcal{O})$ denote its field of fractions. We consider the unique continuous isomorphisms of \mathcal{O} -algebras

$$t_n : \mathcal{O}[[\mathcal{G}]] \xrightarrow{\sim} \mathcal{O}[[\mathcal{G}]], \quad \iota : \mathcal{O}[[\mathcal{G}]] \xrightarrow{\sim} \mathcal{O}[[\mathcal{G}]]^{\mathrm{op}}$$

satisfying $t_n(g) = c_p(g)^n \cdot g$ and $\iota(g) = g^{-1}$, for all $g \in \mathcal{G}$ and all $n \in \mathbb{Z}$. For any $\mathcal{O}[[\mathcal{G}]]$ -module M and any $n \in \mathbb{Z}$, we let $M(n)$ denote the usual Tate twist of M by c_p^n . More precisely, $M(n) := M$ with a new $\mathcal{O}[[\mathcal{G}]]$ -action given by $\lambda * x := t_n(\lambda) \cdot x$, for all $\lambda \in \mathcal{O}[[\mathcal{G}]]$ and $x \in M$. Also, we let $M^* := \mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$ and view it as an $\mathcal{O}[[\mathcal{G}]]$ -module with the contra-variant \mathcal{G} -action given by $g \cdot f(x) = f(\iota(g) \cdot x)$, for all $f \in M^*$, $x \in M$ and $g \in \mathcal{G}$. Throughout, \mathcal{O} and $Q(\mathcal{O})$ are viewed as a $\mathcal{O}[[\mathcal{G}]]$ -modules with the trivial \mathcal{G} -action. Also, if M and N are $\mathcal{O}[[\mathcal{G}]]$ -modules, the $M \otimes_{\mathcal{O}} N$ is viewed as an $\mathcal{O}[[\mathcal{G}]]$ -module with the diagonal \mathcal{G} -action.

Lemma 7.4. *Assume that \mathcal{G} is abelian and M is a finitely presented $\mathcal{O}[[\mathcal{G}]]$ -module. Then, for all $n \in \mathbb{Z}$, the following hold.*

- (1) $\mathrm{Ann}_{\mathcal{O}[[\mathcal{G}]]}(M(n)) = t_{-n}(\mathrm{Ann}_{\mathcal{O}[[\mathcal{G}]]}(M));$
- (2) $\mathrm{Fit}_{\mathcal{O}[[\mathcal{G}]]}(M(n)) = t_{-n}(\mathrm{Fit}_{\mathcal{O}[[\mathcal{G}]]}(M));$
- (3) $\mathrm{pd}_{\mathcal{O}[[\mathcal{G}]]} M = \mathrm{pd}_{\mathcal{O}[[\mathcal{G}]]} M(n);$
- (4) *If $k := \mathbb{F}_q$ is the finite field of q elements ($p \nmid q$), $\sigma_q \in \mathcal{G}$ is the q -power Frobenius automorphism of \mathcal{K} , and $M := T_p(\mathcal{K}^\times)$, then*

$$\mathrm{Fit}_{\mathbb{Z}_p[[\mathcal{G}]]}(M(n)) = \mathrm{Ann}_{\mathbb{Z}_p[[\mathcal{G}]]}(M(n)) = (1 - q^n \cdot \sigma_q^{-1}).$$

Proof. For the easy proof of (1) and (2), see Lemma 3.1 in [23]. Part (3) is an immediate consequence of the existence of a unique $\mathbb{Z}_p[[\mathcal{G}]]$ -module isomorphism $\mathbb{Z}_p[[\mathcal{G}]] \simeq \mathbb{Z}_p[[\mathcal{G}]](n)$ which sends $g \rightarrow c(g)^n g$, for all $g \in \mathcal{G}$. Part (4) is derived from

part (2) as follows. Under the hypotheses of part (4), \mathcal{G} is a pro-cyclic group of (topological) generator σ_q . Since \mathbb{Z}_p is a cyclic $\mathbb{Z}_p[[\mathcal{G}]]$ -module, (43) gives

$$\text{Fit}_{\mathbb{Z}_p[[\mathcal{G}]]}(\mathbb{Z}_p) = \text{Ann}_{\mathbb{Z}_p[[\mathcal{G}]]}(\mathbb{Z}_p) = (1 - \sigma_q^{-1}).$$

Now, note that $T_p(\mathcal{K}^\times) = T_p(\mu_{p^\infty}) = \mathbb{Z}_p(1)$ and $c(\sigma_q) = q$, and apply (1) with $M := \mathbb{Z}_p$. \square

Throughout the rest of this subsection, we will assume that \mathcal{K} is a finite, abelian extension of Galois group G of the cyclotomic \mathbb{Z}_p -extension \mathcal{K}' of k and that \mathcal{G} is abelian. Consequently, there is a non-canonical group isomorphism $\mathcal{G} \simeq G \times \Gamma$, where $\Gamma \simeq \mathbb{Z}_p$. We fix a topological generator γ of Γ . We identify $\mathcal{O}[[\mathcal{G}]]$, $\mathcal{O}[G][[\Gamma]]$ and $\mathcal{O}[G][[t]]$ via the obvious $\mathcal{O}[G]$ -algebra isomorphisms

$$\mathcal{O}[[\mathcal{G}]] \simeq \mathcal{O}[G][[\Gamma]] \simeq \mathcal{O}[G][[t]], \quad \gamma \rightarrow (t+1).$$

We let $K := \mathcal{K}^\Gamma$ and identify G and Γ with $G(K/k)$ and $G(\mathcal{K}'/k)$ via the usual Galois-restriction isomorphisms. Note that since $\mu_{p^\infty} \subseteq \mathcal{K}$, we have $\mu_p \subseteq K$. Consequently, ω and κ factor through G and Γ , respectively. For simplicity, we assume that \mathcal{O} contains the values of all the irreducible \mathbb{C}_p -valued characters of G . We denote by $\widehat{G}(\mathbb{C}_p)$ the set of all \mathbb{C}_p -valued irreducible characters of G and, for all $\chi \in \widehat{G}(\mathbb{C}_p)$, we let $e_\chi := 1/|G| \sum_{\sigma \in G} \chi(\sigma) \cdot \sigma^{-1}$ denote the idempotent associated to χ in $Q(\mathcal{O})[G]$. Each χ as above will be extended to the unique $Q(\mathcal{O})[X]$ -algebra and $Q(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}[[\Gamma]]$ -algebra morphisms

$$\chi : Q(\mathcal{O})[G][X] \rightarrow Q(\mathcal{O})[X], \quad \chi : Q(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}[[\mathcal{G}]] \rightarrow Q(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}[[\Gamma]]$$

which send $g \rightarrow \chi(g)$, for all $g \in G$, respectively. Also, for a polynomial $P \in Q(\mathcal{O})[G][X]$, we denote by $P(\gamma)$ (respectively $P(t+1)$) its image via the unique $Q(\mathcal{O})[G]$ -algebra morphism $Q(\mathcal{O})[G][X] \rightarrow Q(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}[[\mathcal{G}]]$ which sends $X \rightarrow \gamma$ (respectively $X \rightarrow (t+1)$.)

Next, we let \mathcal{L} denote an $\mathcal{O}[[\mathcal{G}]]$ -module, which is free of finite rank as an \mathcal{O} -module. We consider the following $Q(\mathcal{O})$ -vector spaces

$$V := Q(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{L}, \quad V^* := \text{Hom}_{Q(\mathcal{O})}(V, Q(\mathcal{O})) \simeq Q(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{L}^*,$$

endowed with the usual $Q(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}[[\mathcal{G}]]$ -module structures. We consider the following polynomials in $Q(\mathcal{O})[G][X]$ and $Q(\mathcal{O})[X]$, respectively:

$$P_V(X) := \det_{Q(\mathcal{O})[G]}(X - \mathbf{m}_\gamma \mid V), \quad P_{V,\chi}(X) := \chi(P_V(X)) = \det_{Q(\mathcal{O})}(X - \mathbf{m}_\gamma \mid e_\chi V),$$

where \mathbf{m}_γ denotes the automorphism of V and $e_\chi V$ given by multiplication with γ , for all $\chi \in \widehat{G}(\mathbb{C}_p)$. Observe that, for all $\chi \in \widehat{G}(\mathbb{C}_p)$ and all $n \in \mathbb{Z}$, we have

$$(45) \quad \chi(t_n(P_V(\gamma))) = P_{V,\chi\omega^n}(\kappa(\gamma)^n \gamma), \quad \chi((\iota \circ t_n)(P_V(\gamma))) = P_{V,\chi^{-1}\omega^n}(\kappa(\gamma)^n \gamma^{-1}).$$

Lemma 7.5. *The following hold for all $n \in \mathbb{Z}$ and all $\chi \in \widehat{G}(\mathbb{C}_p)$.*

- (1) *If \mathcal{L} is a projective $\mathcal{O}[G]$ -module, then $P_{V(n)}(X)$, $P_{V^*(n)}(X)$ are monic polynomials in $\mathcal{O}[G][X]$.*
- (2) *$P_{V(n),\chi}(X)$, $P_{V^*(n),\chi}$ are monic polynomials in $\mathcal{O}[X]$.*
- (3) *$P_{V(n),\chi}(\gamma) \sim \chi(t_{-n}(P_V(\gamma)))$ and $P_{V^*(n),\chi}(\gamma) \sim \chi((\iota \circ t_n)(P_V(\gamma)))$, where “ \sim ” denotes association in divisibility in the ring $\mathcal{O}[[\Gamma]]$.*

Proof. Fix an $n \in \mathbb{Z}$ and note that we have $Q(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}[[G]]$ -module isomorphisms

$$(46) \quad V(n) \simeq Q(\mathcal{O})[G] \otimes_{\mathcal{O}[G]} \mathcal{L}(n), \quad V^*(n) \simeq Q(\mathcal{O})[G] \otimes_{\mathcal{O}[G]} \mathcal{L}^*(n).$$

In order to prove (1), first note that if \mathcal{L} is $\mathcal{O}[G]$ -projective, then the $\mathcal{O}[G]$ -modules $\mathcal{L}(n) \simeq \mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}(n)$ and $\mathcal{L}^*(n) \simeq \mathcal{L}(-n)^*$ are projective as well. Indeed, since the modules in question are \mathcal{O} -free and \mathcal{O} is a PID, their $\mathcal{O}[G]$ -projectivity is equivalent to their G -cohomological triviality (See [27], Ch. IX, §5, Theorem 7 for $\mathbb{Z}[G]$ -modules. The same proof works for $\mathcal{O}[G]$ -modules, for a general PID \mathcal{O} .) Now, apply the Corollary to Proposition 1 in [27], Ch. IX, §3 to arrive at the desired result. (Loc.cit. deals with the case of $\mathbb{Z}[G]$ -modules. The same argument works for $R[G]$ -modules, where R is a PID, in particular $R = \mathcal{O}$.) Now, part (1) follows by applying (42) to conclude that

$$P_{V(n)}(X) = \det_{\mathcal{O}[G]}(X - \mathbf{m}_\gamma | \mathcal{L}(n)), \quad P_{V^*(n)}(X) = \det_{\mathcal{O}[G]}(X - \mathbf{m}_\gamma | \mathcal{L}^*(n)).$$

The monicity follows from Remark 7.1.

Part (2) follows similarly: First, one uses (46) to conclude that there are isomorphisms of $Q(\mathcal{O}) \otimes_{\mathcal{O}} [[\Gamma]]$ -modules $e_\chi \cdot V(n) \simeq Q(\mathcal{O}) \otimes_{\mathcal{O}} e_\chi \cdot \mathcal{L}(n)$ and $e_\chi \cdot V^*(n) \simeq Q(\mathcal{O}) \otimes_{\mathcal{O}} e_\chi \cdot \mathcal{L}^*(n)$, where $e_\chi \cdot \mathcal{L}(n)$ and $e_\chi \cdot \mathcal{L}^*(n)$ are viewed as (necessarily free) \mathcal{O} -submodules of maximal rank in $e_\chi \cdot V(n)$ and $e_\chi \cdot V^*(n)$, respectively. Then, (42) gives the equalities

$$P_{V(n),\chi}(X) = \det_{\mathcal{O}}(X - \mathbf{m}_\gamma | e_\chi \cdot \mathcal{L}(n)), \quad P_{V^*(n),\chi}(X) = \det_{\mathcal{O}}(X - \mathbf{m}_\gamma | e_\chi \cdot \mathcal{L}^*(n)),$$

which conclude the proof of part (2).

Next, we prove the second “ \sim ” in part (3). The first “ \sim ” is proved similarly. For every $\chi \in \widehat{G}(\mathbb{C}_p)$, we fix an \mathcal{O} -basis \mathbf{x}_χ of $e_\chi \mathcal{L}$. Note that \mathbf{x}_χ is also a $Q(\mathcal{O})$ -basis of $e_\chi V$. Let $A_{\gamma,\chi}$ be the matrix of \mathbf{m}_γ restricted to $e_\chi V$ with respect to this basis. Then, $A_{\gamma,\chi} \in \mathrm{GL}_{m_\chi}(\mathcal{O})$, where $m_\chi = \mathrm{rk}_{\mathcal{O}} e_\chi \mathcal{L} = \dim_{Q(\mathcal{O})} e_\chi V$. It is easily proved that for all n and χ as above, the matrix of \mathbf{m}_γ restricted to $e_\chi \cdot V^*(n) = (e_{\chi^{-1}\omega^n} V)^*$ with respect to the basis $\mathbf{x}_{\chi^{-1}\omega^n}^*$ (dual basis of $\mathbf{x}_{\chi^{-1}\omega^n}$) is $\kappa(\gamma)^n \cdot (A_{\gamma,\chi^{-1}\omega^n}^{-1})^t$, where t stands for transposition. Consequently, (45) combined with the fact that $\det(A_{\gamma,\chi^{-1}\omega^n}) \in \mathcal{O}^\times$ and $\gamma \in \mathcal{O}[[\Gamma]]^\times$ imply that the following hold in $\mathcal{O}[[\Gamma]]$:

$$\begin{aligned} P_{V^*(n),\chi}(\gamma) &= \det(\gamma \cdot I_{m_{\chi^{-1}\omega^n}} - \kappa(\gamma)^n \cdot (A_{\gamma,\chi^{-1}\omega^n}^{-1})^t) \\ &\sim \det(\kappa(\gamma)^n \gamma^{-1} \cdot I_{m_{\chi^{-1}\omega^n}} - A_{\gamma,\chi^{-1}\omega^n}) \\ &= \chi((\iota \circ t_n)(P_V(\gamma))) \end{aligned}$$

This concludes the proof of the Lemma. \square

7.3. Equivariant Power Series. (Compare with §2 in [2].) Let G be an arbitrary finite abelian group, p a prime, and \mathcal{O} a finite, integral extension of \mathbb{Z}_p which contains the values of all characters $\chi \in \widehat{G}(\mathbb{C}_p)$. We let π be a uniformizer of \mathcal{O} . We identify the set of \mathbb{Z}_p -algebra morphisms $\mathrm{Hom}(\mathbb{Z}_p[G], \mathcal{O})$ with the set $\chi \in \widehat{G}(\mathbb{C}_p)$ of \mathbb{C}_p -valued characters of G in the obvious manner. We let I denote a radical ideal of $\mathbb{Z}_p[G]$ of pure codimension 1. It is easily seen that this means that $I = \cap_{\chi \in \mathcal{F}} \ker(\chi)$, for some set $\mathcal{F} \subseteq \mathrm{Hom}(\mathbb{Z}_p[G], \mathcal{O})$. In what follows, we let $\mathfrak{A} := \mathbb{Z}_p[G]/I$. Obviously, \mathcal{F} coincides with the set of \mathbb{Z}_p -algebra morphisms $\mathrm{Hom}(\mathfrak{A}, \mathcal{O})$. Also, we have an injective \mathbb{Z}_p -algebra morphism

$$\mathfrak{A} \longrightarrow \oplus_\chi \mathcal{O}, \quad x \longrightarrow (\chi(x))_{\chi \in \mathrm{Hom}(\mathfrak{A}, \mathcal{O})}.$$

For every $\chi \in \text{Hom}(\mathfrak{A}, \mathcal{O})$, we abuse notation once again and let χ also denote the unique $\mathfrak{A}[[t]]$ -algebra morphism

$$\chi : \mathfrak{A}[[t]] \longrightarrow \mathcal{O}[[t]],$$

which sends $x \rightarrow \chi(x)$, for all $x \in \mathfrak{A}$.

Definition 7.6. (1) The μ -invariant $\mu(f)$ of a power series $f \in \mathcal{O}[[t]]$ is the largest exponent $r \in \mathbb{Z}_{\geq 0}$, such that $f \in \pi^r \mathcal{O}[[t]]$.

(2) A power series $F \in \mathfrak{A}[[t]]$ is said to have μ -invariant equal to 0 (and we write $\mu(F) = 0$) if

$$\mu(\chi(F)) = 0, \quad \text{for all } \chi \in \text{Hom}(\mathfrak{A}, \mathcal{O}).$$

(3) A polynomial $F \in \mathfrak{A}[t]$ is said to be Weierstrass if $\chi(F)$ is a Weierstrass polynomial in $\mathcal{O}[t]$ (i.e. $\chi(F)$ is monic and all its non-leading coefficients are divisible by π), for all χ as above.

Remark 7.7. The rings \mathfrak{A} considered above are the most general reduced quotients of $\mathbb{Z}_p[G]$ of pure Krull dimension 1. It is easy to see that any ring \mathfrak{A} as above is admissible, in the sense of [2], §2. Also, it is easy to prove that our definition of power series $F \in \mathfrak{A}[[t]]$ of μ -invariant equal to 0 is equivalent with the definition in loc.cit., for all rings \mathfrak{A} as above.

In the following, if R is a commutative ring with 1, and $f, g \in R$, we write “ $f \sim g$ in R ” to mean that f and g are associated in divisibility in R , i.e. there exists a unit $u \in R^\times$, such that $f = u \cdot g$.

Lemma 7.8. Assume that $F, \Theta \in \mathfrak{A}[[t]]$, such that $\mu(F) = \mu(\Theta) = 0$ and

$$\chi(F) \sim \chi(\Theta) \text{ in } \mathcal{O}[[t]], \quad \text{for all } \chi \in \text{Hom}(\mathfrak{A}, \mathcal{O}).$$

Then, we have $F \sim \Theta$ in $\mathfrak{A}[[t]]$.

Proof. Since $\mu(F) = \mu(\Theta) = 0$, Proposition 2.1 in [2] (the equivariant Weierstrass preparation theorem) shows that there exist unique Weierstrass polynomials $f, g \in \mathfrak{A}[t]$ and units $u, v \in \mathfrak{A}[[t]]^\times$, such that

$$F = u \cdot f, \quad \Theta = v \cdot \theta.$$

This implies that we have Weierstrass decompositions in $\mathcal{O}[[t]]$

$$\chi(F) = \chi(u) \cdot \chi(f), \quad \chi(\Theta) = \chi(v) \cdot \chi(\theta),$$

with $\chi(u), \chi(v) \in \mathcal{O}[[t]]^\times$ and $\chi(f), \chi(\theta)$ Weierstrass polynomials in $\mathcal{O}[t]$, for all $\chi \in \text{Hom}(\mathfrak{A}, \mathcal{O})$. However, since the Weierstrass decomposition is unique in $\mathcal{O}[[t]]$ (according to the classical Weierstrass preparation theorem), our hypotheses combined with the above equalities imply that

$$\chi(f) = \chi(\theta), \quad \text{for all } \chi \in \text{Hom}(\mathfrak{A}, \mathcal{O}).$$

Consequently, we have $f = \theta$ and $F = uv^{-1} \cdot \Theta$. Therefore, $F \sim \Theta$ in $\mathfrak{A}[[t]]$. \square

Now, let us assume that p is odd and that G has an element j of order 2. Let $\mathbb{Z}_p[G]^- := \mathbb{Z}_p[G]/(1+j)$ and call a character $\chi \in \widehat{G}(\mathbb{C}_p)$ odd if $\chi(j) = -1$. Clearly, $I := (1+j) = \cap_{\chi} \ker(\chi)$, where χ runs through the odd characters of G . Therefore, $\mathfrak{A} := \mathbb{Z}_p[G]^-$ is a reduced quotient of $\mathbb{Z}_p[G]$ of pure Krull dimension 1. Therefore, the following is a direct consequence of the above Lemma.

Corollary 7.9. *Let $F, \Theta \in \mathbb{Z}_p[G]^-[[t]]$, such that*

$$\mu(\chi(F)) = \mu(\chi(\Theta)) = 0, \quad \chi(F) \sim \chi(\Theta) \text{ in } \mathcal{O}[[t]],$$

for all odd $\chi \in \widehat{G}(\mathbb{C}_p)$. Then, we have $F \sim \Theta$ in $\mathbb{Z}_p[G]^-[[t]]$.

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